

EVERY RATIONAL HODGE ISOMETRY BETWEEN TWO $K3$ SURFACES IS ALGEBRAIC

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ABSTRACT. Consider any rational Hodge isometry $\psi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$ between any two Kähler $K3$ surfaces S_1 and S_2 . We prove that the cohomology class of ψ in $H^{2,2}(S_1 \times S_2)$ is a polynomial in Chern classes of coherent analytic sheaves over $S_1 \times S_2$. Consequently, the cohomology class of ψ is algebraic whenever S_1 and S_2 are algebraic.

CONTENTS

1. Introduction	1
2. Mukai's result: classical and new formulations	4
2.1. A new formulation of Mukai's result	5
3. Double orbits	8
3.1. Double orbits in the even rank two unimodular hyperbolic lattice	9
3.2. Double orbits in the $K3$ lattice	11
3.3. An important example	13
4. Moduli spaces of marked Hodge isometric $K3$ s	14
4.1. The twisted period domain	15
4.2. Definition of the moduli spaces	20
5. Hyperholomorphic sheaves	26
6. Proof of the main result	30
6.1. Surjectivity of \bar{p}	30
6.2. Reduction to the case of a cyclic isometry.	31
7. Appendix	33
8. Acknowledgements	37
References	37

1. INTRODUCTION

A $K3$ surface is a simply connected smooth compact complex manifold of complex dimension 2 with a trivial canonical bundle. Let \mathbb{F} be any of the fields $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or the ring \mathbb{Z} . Let S_1, S_2 be $K3$ surfaces. A $K3$ lattice Λ is an even unimodular lattice of signature $(3, 19)$. All such lattices are isomorphic. A typical example of a $K3$ lattice is the second cohomology lattice $H^2(S, \mathbb{Z})$ of a $K3$ surface S with the bilinear form given by the intersection from.

To any Hodge homomorphism $\varphi : H^2(S_1, \mathbb{F}) \rightarrow H^2(S_2, \mathbb{F})$ (of type $(0,0)$) we can associate a cycle

$$\begin{aligned} Z_\varphi &\in H^{2,0}(S_1, \mathbb{C})^* \otimes H^{2,0}(S_2, \mathbb{C}) \oplus H^{0,2}(S_1, \mathbb{C})^* \otimes H^{0,2}(S_2, \mathbb{C}) \oplus H^{1,1}(S_1, \mathbb{C})^* \otimes H^{1,1}(S_2, \mathbb{C}) \\ &\subset H^{2,2}(S_1 \times S_2, \mathbb{C}), \end{aligned}$$

such that

$$Z_\varphi \in H^{2,2}(S_1 \times S_2, \mathbb{C}) \cap H^4(S_1 \times S_2, \mathbb{F}).$$

Here we naturally identify the vector space $H^{p,q}(S_i, \mathbb{C})^*$ with $H^{n-p,n-q}(S_i, \mathbb{C})$. The Torelli theorem for projective $K3$ surfaces, proved by I.R. Shafarevich and I. Piatetski-Shapiro in [27], states that given an *effective* Hodge isometry $\varphi : H^2(S_1, \mathbb{Z}) \rightarrow H^2(S_2, \mathbb{Z})$ of the cohomology lattices of $K3$ surfaces S_1 and S_2 there exists a unique isomorphism $f : S_2 \rightarrow S_1$ inducing φ . So, in particular, the cycle Z_φ , cohomologous to the graph Γ_f of the map f , is algebraic. In his 1970 ICM talk “Le theoreme de Torelli...”, [29] I. R. Shafarevich asked if Z_φ is algebraic for any *rational* Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$. Our goal is to prove the following theorem.

Theorem 1.1. *Let S_1, S_2 be projective $K3$ surfaces. The class Z_φ of any Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$ is algebraic.*

Theorem 1.1 is proved in Section 6.2. In fact, we will prove a generalization of Theorem 1.1 for Kähler $K3$ surfaces, which is formulated below. We say that a cohomology class on a complex-analytic manifold M is *of analytic type* if it can be expressed as a polynomial in Chern classes of coherent sheaves on M . Now the generalization is the following theorem.

Theorem 1.2. *Let S_1 and S_2 be Kähler $K3$ surfaces. For any rational Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$ the class Z_φ is of analytic type.*

Theorem 1.2 is proved in Section 6.2. In case when the Hodge isometry φ is integral the statement that Z_φ is of analytic type follows from the strong Torelli theorem for analytic $K3$ surfaces proved by Burns and Rapoport in [6]. We see that in case S_1 and S_2 are algebraic $K3$ surfaces, Theorem 1.1 follows from Theorem 1.2.

S. Mukai’s groundbreaking work [22] made a significant progress towards this result (see section 2 below). His main result implies Theorem 1.1 in case when the Picard rank of the surfaces S_1 and S_2 is greater or equal 11 (Hodge isometric surfaces obviously have equal Picard rank), see [22, Corollary 1.10]. Nikulin [26] extended Mukai’s argument to the case of projective elliptic surfaces, which by Meyer’s Theorem (Corollary 5.10 in [8]) implies Theorem 1.1 for surfaces with Picard number greater or equal 5. Mukai announced also Theorem 1.1 in his 2002 ICM talk [24].

Let S be a $K3$ surface, let M be a smooth projective 2-dimensional moduli space of slope-stable vector bundles on S and let Q be a *quasiuniversal* sheaf on $S \times M$. Mukai proved that a normalization of the algebraic class $ch(Q)\sqrt{td_{S \times M}} \in \oplus_p H^{p,p}(S \times M, \mathbb{Q})$ induces a Hodge isometry of the rationalized transcendental lattices of S and M . Here $ch(Q)$ as usually denotes the Chern character of Q and $\sqrt{td_S}$ is the square root of the Todd class of the surface S .

In the present work we use a different cohomology class, a κ -class, which is introduced in Section 2, associated now to a *twisted* universal sheaf on $S \times M$ and inducing a Hodge isometry between $H^2(S, \mathbb{Q})$ and $H^2(M, \mathbb{Q})$. See the definition of a

twisted sheaf and the way the κ -class induces the Hodge isometry in subsection 2.1. This Hodge isometry restricts to Mukai's above mentioned isometry of transcendental lattices.

Let us fix a $K3$ lattice Λ . Further we will need the notion of a *marking* of a $K3$ surface S , that is a lattice isometry $\eta_S : H^2(S, \mathbb{Z}) \rightarrow \Lambda$. Let us choose markings η_1, η_2 for the surfaces S_1, S_2 . For the rest of the introduction the notation φ stands for a rational Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$ and ϕ stands for the rational isometry $\phi = \eta_2 \circ \varphi \circ \eta_1^{-1} : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$, where $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$. So, the markings allow us to assign to rational isometries of different surfaces some elements of the group of isometries of $\Lambda_{\mathbb{Q}}$, for which there exists a nice structure theory. The rational isometries of $\Lambda_{\mathbb{Q}}$ induced by (quasi-)universal sheaves are of the so called *cyclic* type which means that it is of the form $g \circ r_x \circ h$, where g, h are integral isometries of Λ and r_x is a reflection of $\Lambda_{\mathbb{Q}}$, $v \mapsto v - \frac{2(v, x)}{(x, x)}x$ for a non-isotropic primitive $x \in \Lambda$. Such a reflection is, in general, defined over \mathbb{Q} , being integral precisely when $(x, x) = \pm 2$, for any even unimodular lattice Λ . We say that the above isometry $g \circ r_x \circ h$ is of *n-cyclic type*¹, if the primitive vector x satisfies $|(x, x)| = 2n$. We will also say that φ is of cyclic type if $\phi = \eta_2 \varphi \eta_1^{-1}$ is of cyclic type. It is immediately clear that this definition does not depend on the choice of markings η_1, η_2 . In Section 3 we prove a lattice-theoretic criterion for an isometry to be of *n-cyclic* type.

First, we prove Theorem 1.2 in the particular case when the Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$ is of cyclic type.

For a general rational Hodge isometry φ , we have a reduction to the cyclic type case. Namely, as is well known, the group of rational isometries of $\Lambda_{\mathbb{Q}}$ is generated by isometries of cyclic type. Representing a general rational isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ as a composition of cyclic ones, $\phi = \phi_k \circ \dots \circ \phi_1$, we then find a sequence of surfaces $S'_1 = S_1, \dots, S'_{k+1} = S_2$ together with their markings η_i so that 1) for each pair S'_i, S'_{i+1} the isometry $\varphi_i = \eta_{i+1}^{-1} \phi_i \eta_i$ is Hodge; 2) each φ_i is induced by a class of analytic type on $S'_i \times S'_{i+1}$. Now taking composition of classes of analytic type as correspondences we get our φ represented by a class of analytic type (this is nontrivial and follows from the Grothendieck-Riemann-Roch theorem and a result of Grauert). This will prove Theorem 1.2 in its most general form. Now, as it was mentioned above, if S_1, S_2 were initially algebraic then any class of analytic type on $S_1 \times S_2$ is, indeed, algebraic and Theorem 1.1 follows.

In order to prove that every rational Hodge isometry φ of cyclic type is of analytic type we need the following four ingredients:

1) For every $n \geq 2$ we need an example of a $K3$ surface S , a two-dimensional smooth and projective moduli space M of stable vector bundles on S and a universal (untwisted) sheaf \mathcal{E} over $S \times M$ inducing via the κ -class a rational Hodge isometry of *n-cyclic* type from $H^2(S, \mathbb{Q})$ to $H^2(M, \mathbb{Q})$, see Conclusion 3.10;

2) Fix a rational isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ of *n-cyclic* type for some $n \geq 2$. Consider the locus in the moduli space of marked pairs $((S_1, \eta_1), (S_2, \eta_2))$ of $K3$ surfaces along which $\varphi = \eta_2^{-1} \circ \phi \circ \eta_1$ stays of Hodge type. We need to show that this locus is covered by twistor lines analogous to twistor lines in the moduli space of marked $K3$ surfaces. The proof of this fact is given in Proposition 4.20;

¹In Definition 3.1 a different definition of *n-cyclic* type will be used, in terms of a criterion which is easier to verify. Both definitions are equivalent by Proposition 3.3.

3) We observe that any two *signed* isometries of n -cyclic type belong to the locus introduced in 2) and are, thus, deformation equivalent (this is proved by Propositions 3.3 and 4.20);

4) M. Verbitsky's result on hyperholomorphic sheaves implying that for every example in 1) the universal sheaf \mathcal{E} over $S \times M$ can be extended as a twisted sheaf over a twistor family containing the hyperkähler manifold $S \times M$, see Theorem 5.2 and the discussion after it. A repeated application of this result will enable us to deform \mathcal{E} to a sheaf on the product $S_1 \times S_2$ for any $((S_1, \eta_1), (S_2, \eta_2))$ in the corresponding locus as in 2).

One of the applications of the main result of this work is the positive solution to the Hodge conjecture for self products of K3 surfaces with complex multiplication, which follows from the solution of the Shafarevich conjecture, see the paper by Ramòn-Mari [28, Theorems 5.1 and 5.4]. Another consequence of Theorem 1.1 is the following: Every rational Hodge isometry $\psi : T(S_1) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow T(S_2) \otimes_{\mathbb{Z}} \mathbb{Q}$, between the transcendental vector spaces of two projective K3 surfaces, is the restriction of an *algebraic* Hodge isometry $\tilde{\psi} : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$. This follows from Witt's Cancellation Theorem, as pointed out in [28, Theorem 5.1].

2. MUKAI'S RESULT: CLASSICAL AND NEW FORMULATIONS

Here we recall the classical formulation of Mukai's result and reformulate it in the form that we will use later in the proof of the main theorem.

Let S be an algebraic K3 surface. In [22] Mukai introduces a weight 2 Hodge structure on the (commutative) cohomology ring $H^*(S, \mathbb{C})$,

$$\tilde{H}^{2,0}(S, \mathbb{C}) = H^{2,0}(S, \mathbb{C}),$$

$$\tilde{H}^{0,2}(S, \mathbb{C}) = H^{0,2}(S, \mathbb{C}),$$

$$\tilde{H}^{1,1}(S, \mathbb{C}) = H^0(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^4(S, \mathbb{C}).$$

Given a class $\alpha \in H^*(S, \mathbb{Z})$ denote by α^i its graded summand in $H^i(S, \mathbb{Z})$. Denote by $\tilde{H}(S, \mathbb{Z})$ the lattice $H^*(S, \mathbb{Z})$ with the integral bilinear Mukai pairing

$$(\alpha, \beta) = -\alpha^4 \beta^0 + \alpha^2 \beta^2 - \alpha^0 \beta^4 \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}.$$

Now let

$$v = (r, \alpha, s) \in \tilde{H}^{1,1}(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^{1,1}(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

be a vector isotropic with respect to Mukai pairing and let h be an ample divisor on S . Recall that $H^{1,1}(S, \mathbb{Z})$ is, by definition, the intersection $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ and assume there exists a compact moduli space $M = M_h(v)$ of vector bundles E on S , slope-stable with respect to h (h -slope-stable), whose Mukai vector $v(E) = ch(E)\sqrt{td_S}$ is equal to v . To formulate Mukai's result we need the definition of a quasiuniversal sheaf on $S \times M$. We will not use this notion except for the classical formulation of Mukai's results. For further details, like existence of quasiuniversal sheaves, we refer to Mukai [22]. A quasiuniversal sheaf \mathcal{F} of similitude $\sigma(\mathcal{F}) \in \mathbb{N}$ is a sheaf on $S \times M$ such that $\mathcal{F}|_{S \times \{m\}} \cong E_m^{\oplus \sigma(\mathcal{F})}$, where E_m is the h -slope-stable vector bundle on S corresponding to $m \in M$. A quasiuniversal sheaf \mathcal{F} on $S \times M$ determines a cycle

$$Z_{\mathcal{F}^*} = ch(\mathcal{F}^*)\sqrt{td_{S \times M}}/\sigma(\mathcal{F}) \in H^*(S \times M, \mathbb{Q}),$$

and hence a Hodge homomorphism

$$f_{\mathcal{F}^*} : \tilde{H}(S, \mathbb{Q}) \rightarrow \tilde{H}(M, \mathbb{Q}),$$

$$\alpha \mapsto \pi_{M*}(\pi_S^*(\alpha) \cdot Z_{\mathcal{F}^*}).$$

Above “ \cdot ” is the standard multiplication in the cohomology ring $H^*(S, \mathbb{Q})$.

Mukai’s observation. Let \mathcal{E} be a quasiuniversal sheaf on $S \times M$ and v^\perp be the orthogonal complement of v in $\tilde{H}(S, \mathbb{Q})$. Mukai observes [22, page 384, Lemma 4.11] that

$$f_{\mathcal{E}^*}(v^\perp) \subset H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q}) \subset \tilde{H}(M, \mathbb{Q})$$

and

$$f_{\mathcal{E}^*}(v) = w,$$

w is the fundamental cocycle in $H^4(M, \mathbb{Z})$.

The properties of $f_{\mathcal{E}^*}$ mentioned above allow to define an induced Hodge homomorphism

$$\tilde{f}_{\mathcal{E}^*} : v^\perp \otimes \mathbb{Q}/\mathbb{Q}v \rightarrow (H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q}))/H^4(M, \mathbb{Q}) \cong H^2(M, \mathbb{Q}) \subset \tilde{H}(M, \mathbb{Q}),$$

For a K3 surface S its transcendental lattice T_S is, by definition, the integral lattice $(H^{1,1}(S, \mathbb{R}) \cap H^2(S, \mathbb{Z}))^\perp$, where the orthogonal complement is taken in $H^2(S, \mathbb{Z})$ with respect to the intersection form. Note that T_S is contained in the orthogonal complement $v^\perp \subset \tilde{H}(S, \mathbb{Z})$ and $T_S \cap \mathbb{Q}v = \{0\}$.

Theorem 2.1. ([22, Theorem 1.5, Theorem A.5]) *There exists a quasiuniversal sheaf \mathcal{E} on $S \times M$ such that the Hodge homomorphism $f_{\mathcal{E}^*}$ induces a Hodge isometry*

$$\tilde{f}_{\mathcal{E}^*} : v^\perp \otimes \mathbb{Q}/\mathbb{Q}v \cong H^2(M, \mathbb{Q}) \subset \tilde{H}(M, \mathbb{Q}).$$

Moreover,

$$(1) \quad \tilde{f}_{\mathcal{E}^*}|_{T_S} : T_S \otimes \mathbb{Q} \rightarrow T_M \otimes \mathbb{Q}$$

is a rational Hodge isometry.

Here the Hodge structure on $v^\perp \otimes \mathbb{Q}/\mathbb{Q}v$ is induced from that of $\tilde{H}(S, \mathbb{Q})$. The Hodge isometry $\tilde{f}_{\mathcal{E}^*}$ does not depend on the choice of the quasiuniversal family \mathcal{E} and is defined over \mathbb{Z} , that is, $\tilde{f}_{\mathcal{E}^*}(v^\perp/\mathbb{Z}v) = H^2(M, \mathbb{Z})$. Note that in general the map $f_{\mathcal{E}^*}$ is not an isometry of the integral Mukai lattices of S and M , but there is an important particular case when it actually is.

Theorem 2.2. ([22, Theorem 4.9]) *If \mathcal{E} is a universal sheaf ($\sigma(\mathcal{E})=1$) then the map $f_{\mathcal{E}^*}$ is a Hodge isometry between $\tilde{H}(S, \mathbb{Z})$ and $\tilde{H}(M, \mathbb{Z})$.*

2.1. A new formulation of Mukai’s result. Generally speaking, the new reformulation is obtained by canonically extending Mukai’s Hodge isometry (1) above to a rational Hodge isometry from $H^2(S, \mathbb{Q})$ to $H^2(M, \mathbb{Q})$. In order to do this we need to introduce a different version of the class $Z_{\mathcal{E}^*}$. The language of quasiuniversal sheaves used by Mukai is not appropriate for us, as sheaves in general do not stay untwisted under deformations, so we need to deal with *twisted sheaves*. For an account of this theory we refer to A. Caldararu’s PhD Thesis, [7]. Recall the definition of a twisted sheaf. Let $\alpha \in C^2(S, \mathcal{O}_S^*)$ be a Čech 2-cocycle in the analytic topology, determined

by an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of S and sections $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_S^*)$. We define an α -twisted sheaf on S to be a pair $(\{\mathcal{F}_i\}_{i \in I}, \{\theta_{ij}\}_{i,j \in I})$ where \mathcal{F}_i are sheaves of \mathcal{O}_S -modules on U_i and $\theta_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \rightarrow \mathcal{F}_i|_{U_i \cap U_j}$ are isomorphisms such that:

- 1) θ_{ii} is the identity for all $i \in I$;
- 2) $\theta_{ij} = \theta_{ji}^{-1}$ for all $i, j \in I$;
- 3) $\theta_{ij} \circ \theta_{jk} \circ \theta_{ki}$ is multiplication by α_{ijk} on $\mathcal{F}_i|_{U_i \cap U_j \cap U_k}$ for all $i, j, k \in I$.

We want to formulate Mukai's result in terms of twisted sheaves. Let \mathcal{E} be a twisted locally free sheaf of rank r on a compact complex manifold X . Then the sheaf $\mathcal{E}^{\otimes r} \otimes \det(\mathcal{E}^*)$ is untwisted. Now we set

$$\kappa(\mathcal{E}) = Sqrt_r(ch(\mathcal{E}^{\otimes r} \otimes \det(\mathcal{E}^*))),$$

where $Sqrt_r(x)$ is equal to $r + \frac{1}{r^r}(x - r^r) + \dots$, the Taylor series of the r -th root function centered at r^r . It is easy to see that when \mathcal{E} is an untwisted sheaf, the class $\kappa(\mathcal{E})$ is equal to $ch(\mathcal{E}) \cdot \exp(-c_1(\mathcal{E})/r)$. The class $\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$ induces a map

$$\varphi_{\mathcal{E}^*} : H^*(S, \mathbb{Q}) \rightarrow H^*(M, \mathbb{Q})$$

in the same fashion as $f_{\mathcal{E}^*}$ defined above. Actually, $\varphi_{\mathcal{E}^*}$ is a rational Hodge homomorphism between $\tilde{H}(S, \mathbb{Q})$ and $\tilde{H}(M, \mathbb{Q})$. Compose this map with the injection $i : H^2(S, \mathbb{Q}) \rightarrow H^*(S, \mathbb{Q}) = H^0(S, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$ and the projection pr_2 of $H^*(M, \mathbb{Q})$ to $H^2(M, \mathbb{Q})$. Denote the resulting map by $\psi_{\mathcal{E}}$ (without $*$),

$$\psi_{\mathcal{E}} = pr_2 \circ \varphi_{\mathcal{E}^*} \circ i.$$

The following is a reformulation of Theorem 2.1 above in the case of a universal (untwisted) sheaf \mathcal{E} . In this case we reformulate Theorem 2.1 in terms of the κ -class of \mathcal{E} . This map is our candidate for the above mentioned canonical extension of $\tilde{f}_{\mathcal{E}^*}$.

Theorem 2.3. *For a universal sheaf \mathcal{E} on $S \times M$ the map $\psi_{\mathcal{E}}$ is a Hodge isometry between $H^2(S, \mathbb{Q})$ and $H^2(M, \mathbb{Q})$.*

The proof of Theorem 2.3 is given in the next subsection.

2.1.1. Comparison of the two formulations. Let us explain why Theorem 2.3 is a reformulation of Theorem 2.1 when \mathcal{E} is a universal sheaf. For this we need to relate $\psi_{\mathcal{E}}$ to $\tilde{f}_{\mathcal{E}^*}$. Assign to the sheaf \mathcal{E} the cohomology class $v(\mathcal{E}^*) = ch(\mathcal{E}^*)\sqrt{td_{S \times M}} \in H^*(S \times M, \mathbb{Q})$. Note that as the sheaf \mathcal{E} is untwisted, the class $\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$ is the same as

$$v(\mathcal{E}^*) \exp(-c_1(\mathcal{E}^*)/r) = \pi_M^* e^{c_1(\mathcal{E}|_{\{s\} \times M})/r} \cdot v(\mathcal{E}^*) \cdot \pi_S^* e^{c_1(\mathcal{E}|_{S \times \{m\}})/r}.$$

We know also that $td_{S \times M} = \pi_S^* td_S \cdot \pi_M^* td_M$ so the projection formula tells us that the rational Hodge homomorphism $\varphi_{\mathcal{E}^*}$ between “rationalized” Mukai lattices $\tilde{H}(S, \mathbb{Q})$ and $\tilde{H}(M, \mathbb{Q})$ is equal to the composition of maps

$$e^{c_1(\mathcal{E}|_{\{s\} \times M})/r} \circ f_{\mathcal{E}^*} \circ e^{c_1(\mathcal{E}|_{S \times \{m\}})/r}.$$

The maps $e^{c_1(\mathcal{E}|_{\{s\} \times M})/r}$ and $e^{c_1(\mathcal{E}|_{S \times \{m\}})/r}$ above are the Hodge isometries of $\tilde{H}(M, \mathbb{Q})$ and $\tilde{H}(S, \mathbb{Q})$ respectively, defined as multiplication by the corresponding class. Denote these classes by e^α and e^β for short. Here is an immediate check that, more

generally, multiplication by $e^\alpha, \alpha \in H^2(S, \mathbb{Q})$, is an isometry of $\tilde{H}(S, \mathbb{Q})$. Take $w \in \tilde{H}(S, \mathbb{Q})$. Then

$$(w \cdot e^\alpha, w \cdot e^\alpha) = -(w \cdot e^\alpha) \cdot (w \cdot e^\alpha)^* = -w \cdot w^* \cdot e^\alpha e^{-\alpha} = (w, w).$$

Above (\cdot, \cdot) is the Mukai pairing. By $*$ we denote the linear operator on $H^*(S, \mathbb{Q})$ reversing the sign of all 2-dimensional elements and acting as identity on 0- and 4-dimensional elements. The check for e^β goes analogously. In our particular case both of these isometries are Hodge homomorphisms of the respective Mukai lattices as $\alpha = c_1(\mathcal{E}|_{S \times \{m\}})/r$ and $\beta = c_1(\mathcal{E}|_{\{s\} \times M})/r$ are elements of $H^{1,1}(S, \mathbb{Q})$ and $H^{1,1}(M, \mathbb{Q})$ respectively. The most essential ingredient in the proof of equivalence of formulations is the following commutative diagram relating $\psi_{\mathcal{E}}$ to $\tilde{f}_{\mathcal{E}}^*$,

$$\begin{array}{ccc}
 H^2(S, \mathbb{Q}) & \xrightarrow{\psi_{\mathcal{E}}} & H^2(M, \mathbb{Q}) \\
 e^\alpha \downarrow & \nearrow pr_2 \circ e^\beta & \parallel \\
 e^\alpha(H^2(S, \mathbb{Q})) & \xrightarrow{f_{\mathcal{E}}^*} f_{\mathcal{E}}^*(e^\alpha(H^2(S, \mathbb{Q}))) \subset \tilde{H}(M, \mathbb{Q}) & \\
 (*) \parallel & & \searrow q \\
 v^\perp \cap (H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})) & & \\
 (**)\downarrow \cong & & \\
 v^\perp/\mathbb{Q}v & \xrightarrow{\tilde{f}_{\mathcal{E}}^*} & H^2(M, \mathbb{Q})
 \end{array}$$

The map q is the quotient map

$$\tilde{H}(M, \mathbb{Q}) = H^0(M, \mathbb{Q}) \oplus H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q}) \rightarrow \tilde{H}(M, \mathbb{Q})/H^4(M, \mathbb{Q}).$$

Below we explain the equality $(*)$, the isomorphism $(**)$ and why q takes $f_{\mathcal{E}}^*(e^\alpha(H^2(S, \mathbb{Q})))$ to $H^2(M, \mathbb{Q}) \subset \tilde{H}(M, \mathbb{Q})/H^4(M, \mathbb{Q})$. Let us give now a detailed analysis of the maps in the diagram. The map $e^\alpha \circ i$ in the composition

$$\psi_{\mathcal{E}} = pr_2 \circ e^\beta \circ f_{\mathcal{E}}^* \circ e^\alpha \circ i : H^2(S, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q}),$$

maps $H^2(S, \mathbb{Q})$ (Hodge) isometrically into $H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$. So for the composition $pr_2 \circ e^\beta \circ f_{\mathcal{E}}^* \circ e^\alpha \circ i$ being a Hodge isometry on $H^2(S, \mathbb{Q})$ is equivalent to $pr_2 \circ e^\beta \circ f_{\mathcal{E}}^*$ being a Hodge isometry on $e^\alpha(H^2(S, \mathbb{Q})) \subset H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$.

Now we want to apply the map $f_{\mathcal{E}}^*$ to $e^\alpha(H^2(S, \mathbb{Q}))$. In order to describe the image we need Mukai's observation above. First, we claim that $e^\alpha(H^2(S, \mathbb{Q}))$ is contained in v^\perp for $v = v(\mathcal{E}^*|_{S \times \{m\}})$ and, moreover, $e^\alpha(H^2(S, \mathbb{Q})) \cap \mathbb{Q}v = \{0\}$. Indeed $e^\alpha(H^2(S, \mathbb{Q})) \perp v$ is equivalent to $H^2(S, \mathbb{Q}) \perp e^{-\alpha} \cdot v$, here \perp is considered in sense of Mukai pairing. But by the definition of e^α its 2-component is $-v^2$, so we see that $e^{-\alpha} \cdot v$ belongs to $H^0(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$, so that $H^2(S, \mathbb{Q}) \perp e^{-\alpha} \cdot v$ holds automatically. Moreover, as $v^0 = r > 0$ we have that

$$(H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})) \cap \mathbb{Q}v = \{0\}.$$

So e^α injects $H^2(S, \mathbb{Q})$ into v^\perp and $f_{\mathcal{E}}^*$ by Mukai's observation above maps $e^\alpha(H^2(S, \mathbb{Q}))$ into $H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q})$. This gives us the explanation for map

q mentioned above. Now, we see that

$$pr_2 \circ e^\beta : H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$$

is a Hodge homomorphism and it preserves the corresponding restriction of Mukai form,

$$(x, y) = (pr_2 \circ e^\beta(x), pr_2 \circ e^\beta(y))$$

for any $x, y \in H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q})$. Finally, for

$$pr_2 \circ e^\beta \circ f_{\mathcal{E}^*} : e^\alpha(H^2(S, \mathbb{Q})) \rightarrow H^2(M, \mathbb{Q})$$

being a Hodge isometry is equivalent to the following two conditions:

1) $f_{\mathcal{E}^*}$ is a Hodge isometry on $e^\alpha(H^2(S, \mathbb{Q})) \subset v^\perp$;

2) image $f_{\mathcal{E}^*} \circ e^\alpha(H^2(S, \mathbb{Q}))$ intersects trivially kernel $\text{Ker } pr_2 \circ e^\beta = H^4(M, \mathbb{Q})$. Let us check that condition 2) actually follows from 1). Condition 1) is equivalent to the statement of Theorem 2.1 about $f_{\mathcal{E}^*}$, via the commutativity of the lower trapezoid in the above diagram. Then for nonzero $x \in e^\alpha(H^2(S, \mathbb{Q}))$, $x = e^\alpha \cdot u$, $u \in H^2(S, \mathbb{Q})$, we may find $v \in H^2(S, \mathbb{Q})$ such that $0 \neq (u, v) = (x, e^\alpha v) = (f_{\mathcal{E}^*}(x), f_{\mathcal{E}^*}(e^\alpha v))$, so $f_{\mathcal{E}^*}(x)$ cannot be in $H^4(M, \mathbb{Q})$ as $H^4(M, \mathbb{Q}) \perp H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q})$.

As v^\perp is of codimension 1 in $\tilde{H}(S, \mathbb{Q})$, $H^2(S, \mathbb{Q})$ and, hence, $e^\alpha(H^2(S, \mathbb{Q}))$ is of codimension 2 in $\tilde{H}(S, \mathbb{Q})$ and, as was mentioned above, $e^\alpha(H^2(S, \mathbb{Q})) \cap \mathbb{Q}v \subset (H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})) \cap \mathbb{Q}v = \{0\}$, we have that

$$v^\perp = e^\alpha(H^2(S, \mathbb{Q})) \oplus \mathbb{Q}v.$$

The last equality explains both (*) and (**). Now, bearing in mind that v is isotropic in sense of Mukai pairing and that

$$f_{\mathcal{E}^*}(\mathbb{Q}v) = H^4(M, \mathbb{Q})$$

where $H^4(M, \mathbb{Q})$ is orthogonal to $H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q})$ again in Mukai sense, we see that $f_{\mathcal{E}^*}$ being a Hodge isometry on $e^\alpha(H^2(S, \mathbb{Q}))$ is equivalent to

$$\tilde{f}_{\mathcal{E}^*} : v^\perp / \mathbb{Q}v \rightarrow H^2(M, \mathbb{Q})$$

being a Hodge isometry. So, map $\psi_{\mathcal{E}}$ is the canonical extension we were looking for, and the equivalence of the old and the new formulation is justified. This actually completes the proof of Theorem 2.3. \square

Indeed Mukai's result can be reformulated in terms of the κ -class in a greater generality, namely it is possible to reformulate it even in the case when the sheaf \mathcal{E} is a twisted sheaf corresponding to a quasiuniversal sheaf Q on $S \times M$. Consider a family T of compact complex manifolds of the form $S_t \times M_t$ containing $S \times M$. Assuming the sheaf \mathcal{E} can be included into a family $\{\mathcal{E}_t\}_{t \in T}$ of (twisted) sheaves on T , the class $\kappa(\mathcal{E}_t^*) \sqrt{td_{S_t \times M_t}} \in H^*(S_t \times M_t, \mathbb{Q})$, $t \in T$, stays of type (2,2) and is a flat section of the local system. Hence, each of these classes induces a Hodge isometry between second rational cohomology of S_t and M_t at every point t of the family T .

3. DOUBLE ORBITS

Let Λ be the $K3$ lattice. In this section we show that for every $n \in \mathbb{Z}^+$ there exists only one double orbit $O(\Lambda)\phi O(\Lambda) \subset O(\Lambda_{\mathbb{Q}})$ where ϕ is a rational isometry of n -cyclic type, a notion which will be introduced below. Here $O(\Lambda)$ and $O(\Lambda_{\mathbb{Q}})$

denote the group of integral isometries of Λ and the group of rational isometries of $\Lambda_{\mathbb{Q}}$ respectively.

Notation: A *lattice* is a free abelian group with a symmetric integral and non-degenerate bilinear pairing. We denote the pairing by (\bullet, \bullet) . Given a lattice L , denote by $L_{\mathbb{Q}}$ the tensor product $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and by $O(L)$ and $O(L_{\mathbb{Q}})$ the corresponding integral and rational isometry groups. Given a sub-lattice M of L of finite index, we regard M^* as a subgroup of $L_{\mathbb{Q}}$ of elements λ , such that (λ, m) is an integer for every element $m \in M$. We then have a flag $M \subset L \subset L^* \subset M^*$.

Given $\phi \in O(L_{\mathbb{Q}})$, set

$$I_{\phi} := L \cap \phi^{-1}(L).$$

Definition 3.1. We say that ϕ is of *n-cyclic type*, if L/I_{ϕ} is a cyclic group of order n .

Example 3.2. Let x be a primitive element of an even unimodular lattice L satisfying $(x, x) = 2d$, for a non-zero integer d . Let $r_x : L_{\mathbb{Q}} \rightarrow L_{\mathbb{Q}}$ be the reflection given by

$$r_x(\lambda) := \lambda - \frac{2(x, \lambda)}{(x, x)}x = \lambda - \frac{(x, \lambda)}{d}x.$$

Then r_x is of $|d|$ -cyclic type, since I_{r_x} is the preimage of the ideal $(d) \subset \mathbb{Z}$ via the surjective homomorphism $(x, \cdot) : L \rightarrow \mathbb{Z}$.

The *double orbit* of an element $\phi \in O(L_{\mathbb{Q}})$ is the subset $O(L)\phi O(L)$. The double orbit of ϕ will be denoted by $[\phi]$. The set of double orbits will be denoted by $O(L) \backslash O(L_{\mathbb{Q}}) / O(L)$. Let us now formulate the most important result of this section.

Proposition 3.3. Let ϕ_1 and ϕ_2 be rational isometries of $\Lambda_{\mathbb{Q}}$ of *n-cyclic type* (in the sense of Definition 3.1). Then

$$[\phi_1] = [\phi_2].$$

The proof of this proposition will be given in Subsection 3.2. We suggest to skip Subsections 3.1 and 3.2 on a first reading, as the proofs there involve lattice-theoretic techniques not used elsewhere in the paper. We would recommend not to skip Subsection 3.3.

3.1. Double orbits in the even rank two unimodular hyperbolic lattice. Let U be the rank 2 even unimodular lattice of signature $(1, 1)$.

Lemma 3.4. Any rational isometry of $U_{\mathbb{Q}}$ is of cyclic type. There exists a one to one correspondence between double orbits of rational isometries of $U_{\mathbb{Q}}$ of *n-cyclic type* with $n > 1$ and pairs (a, b) of integers satisfying $a > b > 0$, $\gcd(a, b) = 1$, and $ab = n$.

Proof. Let $\{e_1, e_2\}$ be a basis of U satisfying $(e_1, e_2) = 1$ and $(e_i, e_i) = 0$, $i = 1, 2$. The four primitive isotropic vectors in U are $\pm e_1$ and $\pm e_2$. The isometry group of U is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and is generated by $-id$ and the isometry interchanging e_1 and e_2 .

Any $f \in O(U_{\mathbb{Q}})$ sends e_1 either to λe_1 or to λe_2 , for some non-zero rational number λ . Post composing with an element of $O(U)$, we may assume that $f(e_1) = \lambda e_1$, for some $\lambda > 0$. Then $f(e_2) = \frac{1}{\lambda} e_2$. Precomposing with an element of $O(U)$ we may

assume that $\lambda > 1$. We can thus write $\lambda = \frac{a}{b}$, where a and b are integers satisfying the conditions in the statement. Then

$$I_f = \{xe_1 + ye_2 : a \text{ divides } y \text{ and } b \text{ divides } x\}.$$

Hence, U/I_f is isomorphic to $(\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z})$, which in turn is isomorphic to $\mathbb{Z}/ab\mathbb{Z}$. \square

Let f be the element of $O(U_{\mathbb{Q}})$ such that $f(e_1) = \frac{a}{b}e_1$, $f(e_2) = \frac{b}{a}e_1$, where $ab = n$ and $\gcd(a, b) = 1$. Note that I_f^*/I_f is isomorphic to $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$, with generators $\{\frac{e_1}{a} + I_f, \frac{e_2}{b} + I_f\}$. Let

$$q : I_f^*/I_f \rightarrow \mathbb{Q}/2\mathbb{Z}$$

be the residual quadratic form. A subgroup of I_f^*/I_f is called *isotropic* if q vanishes on it. A subgroup of I_f^*/I_f is called *lagrangian*, if it is an isotropic cyclic group of order n .

Let \mathcal{D}_n be the set of all ordered pairs of positive integers (c, d) , satisfying $n = cd$ and $\gcd(c, d) = 1$. Given a pair (c, d) in \mathcal{D}_n , let

$$L_{(c,d)} := \left\langle \left(\frac{c}{a}\right) e_1 + I_f, \left(\frac{d}{b}\right) e_2 + I_f \right\rangle$$

be the subgroup of I_f^*/I_f generated by the two cosets above. Note the equalities

$$L_{(a,b)} = U/I_f \quad \text{and} \quad L_{(b,a)} = f^{-1}(U)/I_f.$$

Let \mathcal{L}_n be the set of lagrangian subgroups of I_f^*/I_f .

Lemma 3.5. (1) *The map $(c, d) \mapsto L_{(c,d)}$ is a one-to-one correspondence from the set \mathcal{D}_n onto the set \mathcal{L}_n .*

$$(2) \quad L_{(c_1,d_1)} \cap L_{(c_2,d_2)} = \langle \text{lcm}(c_1, c_2) \frac{e_1}{a} + I_f, \text{lcm}(d_1, d_2) \frac{e_2}{b} + I_f \rangle.$$

$$(3) \quad L_{(c_1,d_1)} \cap L_{(c_2,d_2)} = (0) \text{ if and only if } (c_1, d_1) = (d_2, c_2).$$

Proof. (1) Injectivity is clear. We prove only the surjectivity. Let L be a lagrangian subgroup of I_f^*/I_f and $(x\frac{e_1}{a}, y\frac{e_2}{b})$ a generator of L . The value $2xy/n$ of q on this generator is congruent to 0 modulo $2\mathbb{Z}$. Hence, n divides xy . Set $c := \gcd(x, n)$ and $d := \gcd(y, n)$. Then n divides cd and L maps injectively into the product of the direct sum of the two cyclic subgroups C_1 generated by $x\frac{e_1}{a} + I_f$ and C_2 generated by $y\frac{e_2}{b} + I_f$. The order of C_1 is n/c and the order of C_2 is n/d . Hence, the order n of L divides the order $n^2/(cd)$ of $C_1 \times C_2$. We conclude that $n = cd$ and L is isomorphic to the product of C_1 and C_2 . It follows that $C_1 \times C_2$ is cyclic and so the orders of C_1 and C_2 are relatively prime. Hence, \mathcal{D}_n maps surjectively onto \mathcal{L}_n .

Part (2) is clear.

Part (3) The “if” direction follows immediately from part (2). We prove the “only if” direction. $L_{(c,d)} \cap L_{(\tilde{c}, \tilde{d})} = (0)$ if and only if $\text{lcm}(c_1, c_2) = n$ and $\text{lcm}(d_1, d_2) = n$, if and only if $\frac{c_1 c_2}{\gcd(c_1, c_2)} = n$ and $\frac{d_1 d_2}{\gcd(d_1, d_2)} = n$. Now $c_1 c_2 d_1 d_2 = n^2$. Hence, $\gcd(c_1, c_2) = 1 = \gcd(d_1, d_2)$, $c_1 c_2 = n$, and $d_1 d_2 = n$. The equality $(c_1, d_1) = (d_2, c_2)$ follows. \square

3.2. Double orbits in the $K3$ lattice. We prove Proposition 3.3 in this section.

Let Λ be the $K3$ lattice. Fix a primitive isometric embedding of U in Λ . We get the orthogonal decomposition $\Lambda = U \oplus U^\perp$. Given a rational isometry $f \in O(U_\mathbb{Q})$ we get a rational isometry \tilde{f} in $O(\Lambda_\mathbb{Q})$ by extending f as the identity on U^\perp . Let

$$\epsilon : O(U) \backslash O(U_\mathbb{Q}) / O(U) \longrightarrow O(\Lambda) \backslash O(\Lambda_\mathbb{Q}) / O(\Lambda)$$

be the function sending $[f]$ to $[\tilde{f}]$.

For a general lattice L the subset of double orbits in $O(L) \backslash O(L_\mathbb{Q}) / O(L)$ of cyclic type of order n will be denoted by

$$\mathcal{O}rb(L, n) \subset O(L) \backslash O(L_\mathbb{Q}) / O(L).$$

Lemma 3.6. *The function ϵ restricts to a surjective map*

$$\epsilon : \mathcal{O}rb(U, n) \rightarrow \mathcal{O}rb(\Lambda, n)$$

for every integer n .

Proof. Let ϕ be a rational isometry of $\Lambda_\mathbb{Q}$ of n -cyclic type. Then both Λ/I_ϕ and I_ϕ^*/Λ are cyclic of order n . Choose an element $x \in I_\phi^* \subset \Lambda_\mathbb{Q}$, which maps to a generator of I_ϕ^*/Λ . We have

$$(2) \quad I_\phi = \{\lambda \in \Lambda : (x, \lambda) \text{ is an integer}\}.$$

The element $y := nx$ belongs to Λ . Set $\text{div}(y, \bullet) := \gcd\{(y, \lambda) : \lambda \in \Lambda\}$. Then n and $\text{div}(y, \bullet)$ are relatively prime. We may assume that y is a primitive element of Λ , possibly after adding to x a primitive element x_1 of Λ , which belongs to a unimodular sublattice of Λ orthogonal to x .

The isometry group of Λ acts transitively on the set of primitive elements of self-intersection $2d$, for any integer d . In other words, this set consists of a single $O(\Lambda)$ -orbit. The element $e_1 + de_2$ of U has degree $2d$. Hence, U contains primitive elements of any even degree and there exists an isometry $g \in O(\Lambda)$, such that $g(y)$ belongs to U .

We claim that $g^{-1}(U^\perp)$ is contained in I_ϕ . Indeed, given an element $t \in U^\perp$, we have

$$(x, g^{-1}(t)) = (g(x), t) = (y, t)/n = 0.$$

The claim follows from Equation (2).

The lattice U^\perp is unimodular, and so $\phi(g^{-1}(U^\perp))$ is a unimodular sublattice of Λ . The orthogonal complement

$$T := [\phi(g^{-1}(U^\perp))]^\perp$$

of the latter in Λ is thus isometric to U . There exists a unique $O(\Lambda)$ -orbit of isometric embeddings of U into Λ . Hence, there exists an isometry $h \in O(\Lambda)$, such that $h(T) = U$. Set

$$\psi := h\phi g^{-1}.$$

Then ψ leaves U^\perp invariant and restricts to U^\perp as an integral isometry ψ_2 . It follows that ψ leaves $U_\mathbb{Q}$ invariant and restricts to $U_\mathbb{Q}$ as a rational isometry f . Let $\tilde{\psi}_2 \in O(\Lambda)$ be the extension of ψ_2 via the identity on U . The extension \tilde{f} of f to an element of $O(\Lambda_\mathbb{Q})$ via the identity on U^\perp satisfies the equality

$$\tilde{f} := \tilde{\psi}_2^{-1} h \phi g^{-1}.$$

Hence, the double orbit $[\phi]$ is equal to $\epsilon([f])$. \square

Lemma 3.7. *Let ϕ_1 and ϕ_2 be rational isometries of $\Lambda_{\mathbb{Q}}$ of n -cyclic type. Assume that $I_{\phi_1} = I_{\phi_2}$. Then $O(\Lambda)\phi_1 = O(\Lambda)\phi_2$. In particular, $[\phi_1] = [\phi_2]$.*

Proof. Set $I := I_{\phi_1}$ and $I^* := I_{\phi_1}^*$. Then $I^* = I_{\phi_2}^*$ as well. There exist integral isometries g_i and h_i in $O(\Lambda)$, and a rational isometry f_i in $O(U_{\mathbb{Q}})$, such that $\phi_i = g_i \tilde{f}_i h_i$, by the surjectivity of ϵ . Then $I = I_{\phi_i} = h_i^{-1}(I_{g_i \tilde{f}_i}) = h_i^{-1}(I_{\tilde{f}_i})$. Hence, the isometry h_i descends to an isometry from I^*/I onto $I_{\tilde{f}_i}^*/I_{\tilde{f}_i}$ with respect to their respective residual quadratic forms. Furthermore, $h_i(\Lambda/I) = \Lambda/I_{\tilde{f}_i}$. Note also that $I_{\tilde{f}_i}^*/I_{\tilde{f}_i}$ is naturally isomorphic to $I_{f_i}^*/I_{f_i}$. Hence, Lemma 3.5 applies also to describe the set of Lagrangian subgroups of I^*/I .

Set $L := \Lambda/I$ and $L_i := \phi_i^{-1}(\Lambda)/I$, $i = 1, 2$. Then $L \cap L_i = (0)$, by definition of I_{ϕ_i} . Hence, $L_1 = L_2$, by Lemma 3.5 (3). It follows that $\phi_1^{-1}(\Lambda) = \phi_2^{-1}(\Lambda)$. We conclude that $\phi_1 \phi_2^{-1}$ belongs to $O(\Lambda)$. \square

Lemma 3.8. *Let λ_1 and λ_2 be primitive elements of Λ and $n > 1$ an integer. Set $I_i := (\Lambda + \text{span}_{\mathbb{Z}}\{\lambda_i/n\})^*$, $i = 1, 2$. There exists an integral isometry $g \in O(\Lambda)$, such that $g(I_1) = I_2$, if and only if there exists an integer k , such that $\gcd(k, n) = 1$, and*

$$(3) \quad \frac{(\lambda_1, \lambda_1)}{2} \equiv k^2 \frac{(\lambda_2, \lambda_2)}{2} \quad \text{modulo } n.$$

Proof. Suppose first that such an isometry g exists. Then $g(I_1^*) = I_2^*$. Hence, $g(\lambda_1/n) + \Lambda = k\lambda_2/n + \Lambda$, for some integer k relatively prime to n . Hence, $g(\lambda_1) = k\lambda_2 + n\alpha$, for some α in Λ . Equality (3) follows.

Assume next that Equation (3) holds for an integer k relatively prime to n . It suffices to find $g \in O(\Lambda)$ and $\alpha \in \Lambda$ such that $g(\lambda_1) = k\lambda_2 + n\alpha$. Set $d_i := (\lambda_i, \lambda_i)/2$, $i = 1, 2$. There exist integers x and y satisfying the equation

$$\frac{d_1 - k^2 d_2}{n} = kx + ny,$$

since $\gcd(k, n) = 1$. Let T be the rank 2 lattice with basis $\{t_1, t_2\}$ and Gram matrix

$$\begin{pmatrix} (t_1, t_1) & (t_1, t_2) \\ (t_2, t_1) & (t_2, t_2) \end{pmatrix} = \begin{pmatrix} 2d_2 & x \\ x & 2y \end{pmatrix}.$$

There exists a primitive isometric embedding $\tau : T \hookrightarrow \Lambda$, by a result due to Nikulin, see [18, Lemma 8.1]. There exists also an isometry h , such that $h(\tau(t_1)) = \lambda_2$, since $\tau(t_1)$ and λ_2 are both primitive elements of Λ of the same degree $2d_2$. Hence, we may assume that $\tau(t_1) = \lambda_2$. Set $\alpha := \tau(t_2)$. Then $\beta := k\lambda_2 + n\alpha$ is a primitive element of Λ satisfying $(\beta, \beta) = 2d_1$. Hence, there exists an integral isometry g satisfying $g(\lambda_1) = \beta$. \square

Proof of Proposition 3.3. We may assume, without loss of generality, that $\phi_i = \tilde{f}_i$, where f_i belongs to $O(U_{\mathbb{Q}})$, by Lemma 3.6. We may further assume that $f_1(e_1) = \frac{a}{b}e_1$, $f_1(e_2) = \frac{b}{a}e_2$, $f_2(e_1) = \frac{c}{d}e_1$, $f_2(e_2) = \frac{d}{c}e_2$, for positive integers a, b, c, d satisfying $ab = n = cd$ and $\gcd(a, b) = 1 = \gcd(c, d)$. Note that $I_{f_1}^*/U$ is generated by $\frac{be_1 + ae_2}{n} + U$ and $I_{f_2}^*/U$ is generated by $\frac{de_1 + ce_2}{n} + U$. Set $\lambda_1 := be_1 + ae_2$ and $\lambda_2 := de_1 + ce_2$. Then $(\lambda_1, \lambda_1) = 2n = (\lambda_2, \lambda_2)$. Hence, there exists an isometry $g \in O(\Lambda)$, such that $g(I_{\tilde{f}_1}) = I_{\tilde{f}_2}$, by Lemma 3.8.

Note the equality $g(I_{\tilde{f}_1}) = g(\Lambda \cap \tilde{f}_1^{-1}(\Lambda)) = \Lambda \cap g\tilde{f}_1^{-1}(\Lambda) = I_{\tilde{f}_1 g^{-1}}$. The equality $I_{\tilde{f}_1 g^{-1}} = I_{\tilde{f}_2}$ follows. Hence, $[\tilde{f}_1 g^{-1}] = [\tilde{f}_2]$, by Lemma 3.7. We conclude the desired equality of the double orbits $[\tilde{f}_1]$ and $[\tilde{f}_2]$. \square

3.3. An important example. Let S be a projective K3 surface, $v = (r, \alpha, s) \in H^0(S, \mathbb{Z}) \oplus H^{1,1}(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ be an isotropic primitive vector, h be a v -generic ample divisor on S and $M = M_h(v)$ be the smooth projective moduli space of h -slope stable sheaves with Mukai vector v on S . Let \mathcal{E} be a universal sheaf on $S \times M$. The associated cohomology class

$$\tilde{Z}_{\mathcal{E}^*} := \kappa(\mathcal{E}^*) \sqrt{td_{S \times M}} = \pi_S^* \sqrt{td_S} \cdot \kappa(\mathcal{E}^*) \cdot \pi_M^* \sqrt{td_M}$$

in $H^*(S \times M, \mathbb{Q})$ determines the map $f_{\mathcal{E}^*} : H^*(S, \mathbb{Q}) \rightarrow H^*(M, \mathbb{Q})$, and thus determines the rational isometry

$$\psi_{\mathcal{E}} : H^2(S, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$$

of Theorem 2.3. We show here that $\psi_{\mathcal{E}}$ is of cyclic type. Let L be the cohomology lattice $H^2(S, \mathbb{Z})$ and $I_{\psi_{\mathcal{E}}} = \psi_{\mathcal{E}}^{-1}(H^2(M, \mathbb{Z})) \cap H^2(S, \mathbb{Z})$. We want to show that the quotient $L/I_{\psi_{\mathcal{E}}}$ is a cyclic group. First of all,

$$\sqrt{\pi_S^* td_S} \cdot \sqrt{\pi_M^* td_M} = 1 + \pi_S^* \mathbf{1}_S + \pi_M^* \mathbf{1}_M + \pi_S^* \mathbf{1}_S \cdot \pi_M^* \mathbf{1}_M \in H^*(S \times M, \mathbb{Z}),$$

where $\mathbf{1}_S, \mathbf{1}_M$ are fundamental classes of S and M . It is easy to see that the map $\psi_{\mathcal{E}} : H^2(S, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$ is actually induced by $\kappa_2(\mathcal{E}^*)$ only, so that we may consider this class instead of $\tilde{Z}_{\mathcal{E}^*}$. The class $\kappa_2(\mathcal{E}^*)$ is easily expressed in terms of $ch_2(\mathcal{E}^*)$ and $c_1(\mathcal{E}^*)$ so that it becomes easy to find the domain in $H^2(S, \mathbb{Z})$ where the map $\psi_{\mathcal{E}}$ induced by $\kappa_2(\mathcal{E}^*)$ takes integral values. Indeed, we see that as

$$\kappa_2(\mathcal{E}^*) = ch_2(\mathcal{E}^*) - c_1^2(\mathcal{E}^*)/2r$$

and $ch_2(\mathcal{E}^*)$ is known to be integral, the question reduces to finding the domain where $c_1^2(\mathcal{E}^*)/2r$ takes integral values. The first Chern class of the restriction $c_1(\mathcal{E}^*|_{S \times \{m\}})$ is the 2-component α of the vector v , denote the other first Chern class restriction $c_1(\mathcal{E}^*|_{\{s\} \times M})$ by β . Then

$$c_1^2(\mathcal{E}^*)/2r = (\pi_S^* \alpha + \pi_M^* \beta)^2/2r = \pi_S^* \alpha^2/2r + \pi_S^* \alpha \cdot \pi_M^* \beta/r + \pi_M^* \beta^2/2r.$$

As the vector v is isotropic, $\alpha^2 = 2rs$, so the fact that $\pi_S^* \alpha^2/2r$ is an integral class follows. It is also easy to see that $\pi_M^* \beta^2/2r$ induces zero map on L . On the other hand, the mixed term $\pi_S^* \alpha \cdot \pi_M^* \beta/r$ induces on L the map

$$\gamma \mapsto (\gamma, \alpha)\beta/r.$$

Let us write $\alpha = k \cdot x, k \in \mathbb{Z}, x$ a primitive class in $L = H^2(S, \mathbb{Z})$ and respectively $\beta = j \cdot y, j \in \mathbb{Z}, y$ a primitive class in $H^2(M, \mathbb{Z})$. The subset of L , where $\psi_{\mathcal{E}}$ takes integral values, is the sublattice

$$\begin{aligned} I_{\psi_{\mathcal{E}}} &= \{\gamma \in L \text{ such that } (\gamma, \alpha)j \text{ is divisible by } r\} = \\ &= \left\{ \gamma \in L \text{ such that } (\gamma, x) \text{ is divisible by } \frac{r}{\gcd(jk, r)} \right\} \subset L. \end{aligned}$$

Let us now discuss the size of the quotient $L/I_{\psi_{\mathcal{E}}}$. Let w be an element of L satisfying $(w, x) = 1$ where x is the primitive element of L introduced above. Such

w exists by the unimodularity of L . Let $\xi : L \rightarrow H^2(M, \mathbb{Q})$ be given by $\gamma \mapsto (\gamma, \alpha)\beta/r$. Then $\xi(\gamma) = (\gamma, x)\xi(w)$. Hence, $\gamma - (\gamma, x)w$ belongs to $I_{\psi_{\mathcal{E}}}$, for every $\gamma \in L$. Consequently, L is generated by w and $I_{\psi_{\mathcal{E}}}$, so the quotient $L/I_{\psi_{\mathcal{E}}}$ is a cyclic group. It has order $\frac{r}{\gcd(jk, r)}$.

Conclusion 3.9. *The rational Hodge isometry $\psi_{\mathcal{E}}$ is of $\frac{r}{\gcd(jk, r)}$ -cyclic type.*

Conclusion 3.10. *For any isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ of n -cyclic type there exist marked projective K3 surfaces (S, η_S) and (M, η_M) , where M is a moduli space of rank n sheaves on S , slope-stable with respect to an ample divisor $h \in \text{Pic}(S)$, a universal locally free family \mathcal{E} over $S \times M$ and a rational Hodge isometry $\psi : H^2(S, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$ induced by $\kappa(\mathcal{E})\sqrt{td_{S \times M}}$, such that $\phi = \eta_S^{-1} \circ \psi \circ \eta_M$.*

Proof. Let us first construct S, M and \mathcal{E} and then pick up the markings. Choose an integer $s > 0$ such that $\gcd(n, s) = 1$. Set $d = sn$. Choose now a K3 surface S so that S is a general member of the family \mathcal{F}_{sn+1} of (quasi-) polarized K3-surfaces of degree $2sn$ introduced in Mukai [23]. Then, in particular, the Picard group of S is cyclic, $\text{Pic}(S) = \mathbb{Z}h$.

Consider the Mukai vector $v = (n, h, s) \in \tilde{H}^2(S, \mathbb{Q})$ and note that $(v, v) = 0$. Take $M = M_h(v)$ to be the moduli space of rank n sheaves on S that are h -Gieseker-stable. As $\text{Pic}(S)$ is cyclic and for every $\mathcal{F}_m, m \in M$, we have $c_1(\mathcal{F}) = h$, the notions of h -slope-stability and h -Gieseker-stability on the surface S are equivalent. Indeed, on a surface slope-stability implies Gieseker stability and Gieseker stability implies slope-semistability. Now $\mu_h(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot h}{rk(\mathcal{F})} > 0$ and the cyclic Picard group condition gives us that for any subsheaf $\mathcal{F}_1 \subset \mathcal{F}$ we have $\mu_h(\mathcal{F}_1) = q \cdot \mu_h(\mathcal{F})$ for some rational $q \in \mathbb{Q}$, $|q| > 1$. Now it is easy to see that there are no h -slope-semistable sheaves that are not h -slope-stable. So indeed M is a moduli space of h -slope-stable sheaves on S .

The manifold M is a K3 surface rational Hodge isometric to S , with Picard group $\mathbb{Z}\hat{h}$ for a primitive ample divisor $\hat{h} \in \text{Pic}(M)$, $\hat{h}^2 = h^2$, see [23, Proposition 1.1]. There exists a universal sheaf \mathcal{E}_0 on $S \times M$, which follows from our choice of $v = (n, h, s)$ and [22, Theorem A.6, Remark A.7]. One can construct from \mathcal{E}_0 a *normalized universal* sheaf \mathcal{E} on $S \times M$ such that $c_1(\mathcal{E}) = \pi_S^* h + k\pi_M^* \hat{h}$ for an appropriate k , $sk \equiv 1 \pmod{n}$. Mukai proved that the sheaf \mathcal{E} is locally free. For the proofs and constructions we refer to his paper [23, Theorems 1.1, 1.2]. Now we refer to Conclusion 3.9, setting $r = n$, $j = 1$ and k being just our k . According to the Conclusion $\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$ induces a Hodge isometry of n -cyclic type, so take ψ to be this isometry.

Let us show the existence of the required markings η_S, η_M . For this we will need Proposition 3.3 stating that for every integer n there exists precisely one *double orbit* of an isometry of n -cyclic type. Take any markings η_1, η_2 of the surfaces S and M . Then the rational cyclic isometry $\phi_1 = \eta_2 \circ \psi \circ \eta_1^{-1} : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ is also of n -cyclic type and thus, by Proposition 3.3, determines the same double orbit $[\phi_1]$ as ϕ does, $[\phi_1] = [\phi]$. From here we see that we can pick up η_S, η_M that take ψ to ϕ as required above. \square

4. MODULI SPACES OF MARKED HODGE ISOMETRIC K3S

In Subsection 4.1 we introduce the notion of a twisted period domain. In Subsection 4.2 we introduce two moduli spaces. One moduli space parametrizes, roughly

speaking, pairs of $K3$ surfaces S_1, S_2 together with a holomorphic vector bundle \mathcal{E} on their product $S_1 \times S_2$. The other moduli space parametrizes, roughly, pairs of $K3$ surfaces S_1, S_2 together with a Hodge isometry $\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$.

4.1. The twisted period domain.

4.1.1. *Marked $K3$ surfaces and their moduli space.* The $K3$ lattice is the unique up to isomorphism even unimodular lattice of signature $(3, 19)$. Fix such a lattice Λ and denote by $q(\cdot, \cdot)$ the corresponding bilinear form. Introduce the notation $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$ for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} . The vector space $\Lambda_{\mathbb{F}}$ is equipped with a bilinear pairing determined by $q(\cdot, \cdot)$. We will denote this pairing by $q(\cdot, \cdot)$ as well. By $[x]$ we denote the class of a vector $x \in \Lambda_{\mathbb{C}}$ in the projective space $\mathbb{P}\Lambda_{\mathbb{C}}$. Let

$$\Omega_{\Lambda} = \{[x] \in \mathbb{P}\Lambda_{\mathbb{C}} \text{ such that } q(x, x) = 0 \text{ and } q(x, \bar{x}) > 0\}$$

be the period domain of $K3$ surfaces, constructed from Λ . Below by σ_S we denote a global holomorphic form on a $K3$ surface S , the letter η with or without subscript stands for a marking of a $K3$ surface, that is, an isometry $\eta : H^2(S, \mathbb{Z}) \rightarrow \Lambda$. We will use the same notation η for both the isometry $\eta : H^2(S, \mathbb{Z}) \rightarrow \Lambda$ and the isomorphism of vector spaces $H^2(S, \mathbb{F}) \rightarrow \Lambda_{\mathbb{F}}$ induced by η . Denote by \mathfrak{M} the moduli space of marked $K3$ surfaces and let

$$\begin{aligned} \pi : \mathfrak{M} &\rightarrow \Omega_{\Lambda}, \\ (S, \eta) &\mapsto [\eta(\sigma_S)], \end{aligned}$$

be the period map.

Let us fix a rational isometry ϕ of $\Lambda_{\mathbb{Q}}$. Then ϕ induces in a natural way an isomorphism $\tilde{\phi} : \Omega_{\Lambda} \rightarrow \Omega_{\Lambda}$. The graph of $\tilde{\phi}$ is the following subset of the self-product of Ω_{Λ} :

$$\Omega_{\phi} = \{(l, \tilde{\phi}(l)) \mid l \in \Omega_{\Lambda}\} \subset \Omega_{\Lambda} \times \Omega_{\Lambda}.$$

The set Ω_{ϕ} serves as a period domain for pairs $(S_1 \times S_2, \eta)$ of products of $K3$ surfaces S_1, S_2 together with a pair of markings

$$\eta = (\eta_1, \eta_2), \eta_i : H^2(S_i, \mathbb{Z}) \rightarrow \Lambda_{\mathbb{Z}},$$

$i = 1, 2$, satisfying the condition that

$$\psi = \eta_2^{-1} \circ \phi \circ \eta_1 : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$$

is a Hodge isometry.

Remark 4.1. For a general complex manifold S we say that a marking is an isomorphism of abelian groups $\eta_S : H^*(S, \mathbb{Z}) \rightarrow L$, where L is some abelian group. For a manifold $S = S_1 \times S_2$, where S_1, S_2 are $K3$ surfaces, we want to consider only markings η_S determined by (isometric) markings η_1, η_2 of S_1 and S_2 via Künneth decomposition of $H^*(S_1 \times S_2, \mathbb{Z})$. The most interesting part for us, of $H^*(S_1 \times S_2, \mathbb{Z})$, is concentrated in degrees 2 and 4. The interesting pieces of degrees 2 and 4 are $H^2(S_1, \mathbb{Z}) \oplus H^2(S_2, \mathbb{Z})$ and $H^2(S_1, \mathbb{Z}) \otimes H^2(S_2, \mathbb{Z})$ respectively. So, by a marking of S we mean an isomorphism $\eta_1 \oplus \eta_2$ or $\eta_1 \otimes \eta_2$, depending on the degree of cohomology classes under consideration.

4.1.2. *Hyperkähler manifolds.* Let us recall the notion of a twistor line in Ω_Λ . Fix a $K3$ surface S and a marking $\eta : H^2(S, \mathbb{Z}) \rightarrow \Lambda$. Take any positive 3-dimensional subspace V in $H^2(S, \mathbb{R})$ containing the plane $P = \langle \text{Re } \sigma_S, \text{Im } \sigma_S \rangle$, and consider the image $\mathbb{P}\eta(V_\mathbb{C})$ of $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ in $\mathbb{P}\Lambda_\mathbb{C}$ under the natural projection. The intersection Q_V of $\mathbb{P}\eta(V_\mathbb{C})$ with $\Omega_\Lambda \subset \mathbb{P}\Lambda_\mathbb{C}$ is a smooth complete conic in the plane $\mathbb{P}\eta(V_\mathbb{C})$, which is called *a twistor line through the point $[\eta(\sigma_S)]$ corresponding to the subspace V* . For a detailed account of theory of twistor lines we refer to [15]. Note that V decomposes as an orthogonal sum $\mathbb{R}\alpha \oplus \langle \text{Re } \sigma_S, \text{Im } \sigma_S \rangle$ for some $\alpha \in H^{1,1}(S, \mathbb{R})$ such that $(\alpha, \alpha) > 0$. We will record this information denoting Q_V by $Q_{S,\alpha}$. There is an important case when α belongs to the Kähler cone of S . In this case it is possible to associate to the twistor line a hyperkähler structure on S . Let us describe this structure. Recall (see [10, p. 548]) that a manifold M is called *hyperkähler* with respect to a metric g if there exist covariantly constant complex structures I, J and K which satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = -1, IJ = -JI = K.$$

We call the ordered triple I, J, K *a hyperkähler structure on M compatible with g* . A hyperkähler structure I, J, K gives rise to a sphere S^2 of complex structures

$$S^2 = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}.$$

Let us consider the case when our hyperkähler manifold is a $K3$ surface S . For any Kähler class on S with a complex structure I there exists a closed $(1,1)$ -form α representing this class and a unique hyperkähler metric g such that α is a class in $H_{DR}^2(S, \mathbb{R})$ represented by a closed $(1,1)$ -form $g(\cdot, I\cdot)$, see [4] and [3, Ch. VIII]. This hyperkähler metric gives rise to hyperkähler structures of the form I, J, K where I is our original complex structure on S . Let us choose such a hyperkähler structure I, J, K . The choice of (J, K) is not unique, any other choice (J', K') is of the form

$$J' = \cos t \cdot J + \sin t \cdot K, K' = -\sin t \cdot J + \cos t \cdot K,$$

for an appropriate $t \in [0, 2\pi]$. This can be seen as follows. Any complex structure $\lambda \in S^2$ determines a Kähler class ω_λ on (S, I) represented by a closed positive $(1,1)$ -form $g(\cdot, \lambda\cdot)$. It follows (see [10, p. 550]) that $\omega_J + i\omega_K$ is the class of a global holomorphic 2-form in $H^2(S, \mathbb{C})$ and is, thus, proportional to σ_S over \mathbb{C} . So all such possible choices of J and K form a $U(1)$ -orbit under the natural action of $U(1)$.

Remark 4.2. For the classes $\omega_I, \omega_J, \omega_K$ from the orthogonality relation $\omega_I \perp \omega_J + i\omega_K$ we obviously have that $\omega_I \cdot \omega_J = \omega_I \cdot \omega_K = 0$. The fact that $\omega_J + i\omega_K$ is a class of a holomorphic 2-form means that it is isotropic, $(\omega_J + i\omega_K) \cdot (\omega_J + i\omega_K) = 0$, which implies that $\omega_J \cdot \omega_J = \omega_K \cdot \omega_K$ and $\omega_J \cdot \omega_K = 0$. One can actually see that all $\omega_I, \omega_J, \omega_K$ have equal length: $\omega_I \cdot \omega_I = \omega_J \cdot \omega_J = \omega_K \cdot \omega_K$. This can be seen either from the discussion before Proposition 13.3 in [3, Ch. VIII] or directly from the fact that if we start with the Kähler class ω_J on (S, J) then the complex structures K and I can be obtained in the way described above, so the cyclic permutation of our triple of complex structures indeed shows that $\omega_K \cdot \omega_K = \omega_J \cdot \omega_J$. This shows that $\omega_I, \omega_J, \omega_K$ is an orthogonal basis of the corresponding 3-space with basis vectors having equal

length. Moreover, it is clear that for all $\lambda = aI + bJ + cK \in S^2$ the classes

$$(4) \quad \omega_\lambda = a\omega_I + b\omega_J + c\omega_K$$

all have equal length.

Now, we have an isomorphism of the space $V = \langle \alpha, \operatorname{Re} \sigma_S, \operatorname{Im} \sigma_S \rangle = \langle \omega_I, \omega_J, \omega_K \rangle$ with the vector subspace $\langle I, J, K \rangle$ spanned by I, J, K in the vector space of global sections of $\operatorname{End}(TS)$. The isomorphism is given by

$$I \mapsto \omega_I, J \mapsto \omega_J, K \mapsto \omega_K.$$

Thus, the vector space $\langle I, J, K \rangle$ inherits the intersection form from V and by Remark 4.2 the basis I, J, K is an orthogonal basis of our space. Thus, the sphere S^2 introduced above is a round sphere in $\langle I, J, K \rangle$, centered at the origin, and there is a natural action of the group $SO(V) \cong SO(3)$ on S^2 .

Definition 4.3. We say that two hyperkähler structures on a $K3$ surface S are *equivalent* if they belong to the same $SO(V)$ -orbit.

As we pointed out above, on a $K3$ surface (S, I) there is a unique equivalence class of hyperkähler structures of the form I, J, K , as all of them lie in the same $U(1)$ -orbit, $U(1) \subset SO(V)$. Note that this is not the case for general hyperkähler manifolds, for example, for products of $K3$ surfaces, which will be figured out later.

The sphere of complex structures S^2 can be identified with \mathbb{P}^1 as explained in [10, p. 554]. Regarding the identification $S^2 \cong \mathbb{P}^1$, the family $\{(S, \lambda)\}_{\lambda \in S^2}$ over the base $S^2 \cong \mathbb{P}^1$ with the almost-complex structure induced in the horizontal direction by that of \mathbb{P}^1 and in the vertical “fiber” direction by λ , is a complex-analytic family (see [10]) and so, by the definition of the moduli space of marked $K3$ surfaces \mathfrak{M} it determines a (holomorphic) classifying map $S^2 \cong \mathbb{P}^1 \rightarrow \mathfrak{M}$. The image of \mathbb{P}^1 in \mathfrak{M} under this map is the lift $\mathbb{P}_{S, \alpha}$ of the conic $Q_{S, \alpha}$ with respect to $\pi : \mathfrak{M} \rightarrow \Omega_\Lambda$ through the point $(S, \eta) \in \mathfrak{M}$. We call such $\mathbb{P}_{S, \alpha} \subset \mathfrak{M}$ (respectively $Q_{S, \alpha} \subset \Omega_\Lambda$) a *twistor line associated to a hyperkähler structure* or a *hyperkähler line*.

Remark 4.4. For every class $[\sigma_\lambda] \in \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}\eta(V_{\mathbb{C}})$ we have a Kähler class $\omega_\lambda \in H^{1,1}(S, \mathbb{R})$ so that $(\omega_\lambda, \operatorname{Re} \sigma_\lambda, \operatorname{Im} \sigma_\lambda)$ is a positively oriented orthogonal basis of $V \subset H^2(S, \mathbb{R})$. Thus, the class $[\sigma_\lambda]$ together with an orientation of V determines the Kähler class ω_λ up to multiplication by a positive scalar. So every twistor family $\mathcal{Y} \rightarrow S^2 \cong Q_{S, \alpha}$ over a hyperkähler line $Q_{S, \alpha}$ is a family of polarized $K3$ surfaces.

Remark 4.5. Note that a twistor line in Ω_Λ through a point $[\sigma_S]$, for S such that $\operatorname{Pic}(S)$ is trivial, is always a line associated to a hyperkähler structure on S . This is due to the fact that for such surfaces the Kähler cone is equal to the positive cone. For this see Huybrechts, [14] or [11, Proposition 5.4], or Verbitsky, [31, Section 6].

4.1.3. Generalities on hyperkähler structures on products of $K3$'s. Choose a positive 3-subspace $V \subset H^2(S_1, \mathbb{R})$ containing $\langle \operatorname{Re} \sigma_{S_1}, \operatorname{Im} \sigma_{S_1} \rangle$. The projectivization $\mathbb{P}\eta_1(V_{\mathbb{C}})$ is a 2-plane in $\mathbb{P}\Lambda_{\mathbb{C}}$ and its intersection with the open subset of the quadric Q determined by $q(\cdot, \cdot)$ is a conic Q_V , namely, the twistor line through the point $[\eta_1(\sigma_{S_1})]$.

Definition 4.6. The *twistor line* $Q_{\phi, V}$ in Ω_ϕ through $([\eta_1(\sigma_{S_1})], [\eta_2(\sigma_{S_2})])$ is the graph $\Gamma_{\tilde{\phi}}$ of the map

$$\tilde{\phi} : Q_V \rightarrow \Omega_\Lambda,$$

$$[\eta_1(\sigma_\lambda)] \mapsto [\phi(\eta_1(\sigma_\lambda))], \lambda \in S^2.$$

The fact that the period of $(S_1 \times S_2, (\eta_1, \eta_2))$ belongs to Ω_ϕ means that $\psi = \eta_2^{-1} \circ \phi \circ \eta_1$ is a Hodge isometry and $\tilde{\phi}([\eta_1(\sigma_\lambda)]) = [\eta_2(\psi(\sigma_\lambda))]$.

Definition 4.7. A *twistor path* in Ω_ϕ consists of a finite ordered sequence Q_1, Q_2, \dots, Q_k of twistor lines in Ω_ϕ , such that every pair of consecutive lines intersect, together with a choice of a point of intersection $q_i \in Q_i \cap Q_{i+1}$.

Consider the twistor line $Q_{S_1, h}$ in Ω_Λ through $[\eta_1(\sigma_{S_1})]$ which is determined by a Kähler class $h \in H^{1,1}(S_1, \mathbb{R})$. We continue to set $\psi := \eta_2^{-1} \circ \phi \circ \eta_1$. Assume that $\psi(h)$ is a Kähler class in $H^2(S_2, \mathbb{R})$. Then the corresponding twistor line $Q_{\psi, h} \subset \Omega_\phi$ will be denoted by $Q_{\psi, h}$. The classes $h, \psi(h)$ and the original complex structures I_1 on S_1 and I_2 on S_2 determine hyperkähler metrics g_1 and g_2 on S_1 and S_2 . Choose hyperkähler structures J'_l, K'_l compatible with metrics $g_l, l = 1, 2$. We get a quaternionic structure $I := I_1 \oplus I_2, J' := J'_1 \oplus J'_2, K' := K'_1 \oplus K'_2$ on $S_1 \times S_2$ compatible with the metric $g := g_1 \oplus g_2$. Again, like above the choice of I, J', K' is not unique, so all such quaternionic structures with I fixed compatible with g form a $U(1) \times U(1)$ -torsor \mathcal{T} consisting of pairs (J, K) of the form

$$\begin{aligned} J &= (\cos t \cdot J'_1 + \sin t \cdot K'_1) \oplus (\cos s \cdot J'_2 + \sin s \cdot K'_2), \\ K &= (-\sin t \cdot J'_1 + \cos t \cdot K'_1) \oplus (-\sin s \cdot J'_2 + \cos s \cdot K'_2), \end{aligned}$$

where $t, s \in [0, 2\pi]$. Given $a \in U(1) \times U(1)$, set $(J_a, K_a) := a(J', K')$. Let $D \subset U(1) \times U(1)$ be the diagonal subgroup.

Observation 4.8. Given $(a_1, a_2) \in U(1) \times U(1)$, the quaternionic structures (I, J_{a_1}, K_{a_1}) and (I, J_{a_2}, K_{a_2}) are equivalent (Definition 4.3), if and only if $a_1 a_2^{-1}$ belongs to D .

Given $t := (J, K) \in \mathcal{T}$, denote by $\pi : \mathcal{Y}_t \rightarrow \mathbb{P}_t^1$ the twistor family corresponding to the hyperkähler structure (I, J, K) on $S_1 \times S_2$. The markings η_1, η_2 determine a period map $\pi_t : \mathbb{P}_t^1 \rightarrow \Omega_\Lambda \times \Omega_\Lambda$.

Lemma 4.9. *There exists a unique equivalence class of hyperkähler structures in \mathcal{T}/D , such that for $t := (J, K)$ in the corresponding D -orbit in \mathcal{T} the period map π_t of the twistor family $\mathcal{Y}_t \rightarrow \mathbb{P}_t^1$ sends \mathbb{P}_t^1 isomorphically onto $Q_{\psi, h} \subset \Omega_\phi$.*

Proof. In order to show the existence we need to find a hyperkähler structure

$$(I, J, K) = (I_1 \oplus I_2, J_1 \oplus J_2, K_1 \oplus K_2)$$

such that for every $\lambda = (\lambda_1, \lambda_2) = (aI_1 + bJ_1 + cK_1, aI_2 + bJ_2 + cK_2)$ in the corresponding sphere S^2 we have that

$$\psi : H^2(S_{1, \lambda_1}, \mathbb{C}) \rightarrow H^2(S_{2, \lambda_2}, \mathbb{C})$$

is a Hodge isometry. This is precisely the condition that $(\pi_t((S_{1, \lambda_1}, \eta_1)), \pi_t((S_{2, \lambda_2}, \eta_2)))$ belongs to $Q_{\psi, h}$. The condition that

$$\psi(H^{2,0}(S_{1, \lambda_1}, \mathbb{C})) = H^{2,0}(S_{2, \lambda_2}, \mathbb{C}),$$

is equivalent to the condition that for $\lambda = (\lambda_1, \lambda_2) \in S^2$ one has

$$(5) \quad \psi(\omega_{\lambda_1}) = \omega_{\lambda_2},$$

where ω_{λ_1} and ω_{λ_2} are Kähler classes on (S_1, λ_1) and (S_2, λ_2) defined by Equation (4). Let us explain this. Let us endow the positive 3-spaces $\langle h, \text{Re } \sigma_{S_1, I_1}, \text{Im } \sigma_{S_1, I_1} \rangle \subset H^2(S_1, \mathbb{R})$ and $\langle \psi(h), \text{Re } \sigma_{S_2, I_2}, \text{Im } \sigma_{S_2, I_2} \rangle \subset H^2(S_2, \mathbb{R})$ with the orientations given by the corresponding bases. Then the Hodge isometry ψ induces an orientation preserving isometry between orthogonal sums

$$\langle h, \text{Re } \sigma_{S_1, I_1}, \text{Im } \sigma_{S_1, I_1} \rangle = \mathbb{R}\omega_{\lambda_1} \oplus \langle \text{Re } \sigma_{S_1, \lambda_1}, \text{Im } \sigma_{S_1, \lambda_1} \rangle$$

and

$$\langle \psi(h), \text{Re } \sigma_{S_2, I_2}, \text{Im } \sigma_{S_2, I_2} \rangle = \mathbb{R}\omega_{\lambda_2} \oplus \langle \text{Re } \sigma_{S_2, \lambda_2}, \text{Im } \sigma_{S_2, \lambda_2} \rangle.$$

We see now that the isometry $\psi : H^2(S_{1, \lambda_1}, \mathbb{C}) \rightarrow H^2(S_{2, \lambda_2}, \mathbb{C})$ is a Hodge isometry precisely when it takes the class of a holomorphic form to the class of a holomorphic form, that is, $\psi(\omega_{J_1} + i\omega_{K_1}) = z \cdot (\omega_{J_2} + i\omega_{K_2})$ for an appropriate $z \in \mathbb{C}$. This is equivalent to isometry ψ being an orientation preserving map between the 2-planes $\langle \text{Re } \sigma_{S_1, I_1}, \text{Im } \sigma_{S_1, I_1} \rangle$ and $\langle \text{Re } \sigma_{S_2, I_2}, \text{Im } \sigma_{S_2, I_2} \rangle$, and so the restriction

$$\psi : \mathbb{R}\omega_{\lambda_1} \rightarrow \mathbb{R}\omega_{\lambda_2}$$

is an orientation preserving isometry as well. So we should have $\psi(\omega_{\lambda_1}) = \omega_{\lambda_2}$ which is Equation (5). Reversing the arguments we get that ψ satisfying Equation (5) for $(\lambda_1, \lambda_2) \in S^2$ is a Hodge isometry between $H^2(S_{1, \lambda_1}, \mathbb{C})$ and $H^2(S_{2, \lambda_2}, \mathbb{C})$.

Now having fixed our hyperkähler structure (I_1, J_1, K_1) on S_1 let us find a hyperkähler structure (I_2, J_2, K_2) on S_2 (I_2 is the original structure on S_2) such that ψ satisfies Equation (5). We have for granted that $\psi(\omega_{I_1}) = \omega_{I_2}$, so it suffices for us to find J_2 and K_2 such that $\psi(\omega_{J_1}) = \omega_{J_2}$ and $\psi(\omega_{K_1}) = \omega_{K_2}$. Fix some complex structures J'_2 and K'_2 on S_2 such that the triple (I_2, J'_2, K'_2) is a hyperkähler structure. Then, as ψ is a Hodge isometry, we have that $\psi(\omega_{J_1} + i\omega_{K_1}) = e^{-it}(\omega_{J'_2} + i\omega_{K'_2})$ for an appropriate $t \in [0, 2\pi]$ and thus we may take $J_2 = \cos t \cdot J'_2 + \sin t \cdot K'_2$ and $K_2 = -\sin t \cdot J'_2 + \cos t \cdot K'_2$. So, a fixed choice of I_1, J_1, K_1 determines in a unique way the choice of I_2, J_2, K_2 . This shows that all such choices of hyperkähler structures for $(S_1 \times S_2, I_1 \oplus I_2)$ and $(h, \psi(h))$ are in the same D -orbit and thus represent the same class in \mathcal{T}/D . \square

Definition 4.10. The image of the base $Q_{\psi, h}$ of the family $(\pi_t, \pi_t) : \mathcal{Y} \rightarrow Q_{\psi, h} \cong \mathbb{P}^1$ in $\mathfrak{M} \times \mathfrak{M}$ under the classifying map is called the *twistor line through* $((S_1, \eta_1), (S_2, \eta_2)) \in \mathfrak{M} \times \mathfrak{M}$ determined by ψ and h and is denoted by

$$(6) \quad \mathbb{P}_{\psi, h}.$$

The line $\mathbb{P}_{\psi, h} \subset \mathfrak{M} \times \mathfrak{M}$ is a lift of $Q_{\psi, h} \subset \Omega_\phi \subset \Omega_\Lambda \times \Omega_\Lambda$ to $\mathfrak{M} \times \mathfrak{M}$ with respect to $(\pi, \pi) : \mathfrak{M} \times \mathfrak{M} \rightarrow \Omega_\Lambda \times \Omega_\Lambda$, so that $\mathbb{P}_{\psi, h} \subset (\pi, \pi)^{-1}(\Omega_\phi)$. Lemma 4.9 tells us that any line of the form $\mathbb{P}_{\psi, h}$ is associated to a hyperkähler structure (on $S_1 \times S_2$)).

Note that comparing to the case of twistor lines in \mathfrak{M} , which may or may not be associated to hyperkähler structures on $K3$ surfaces, we here define a twistor line on $\mathfrak{M} \times \mathfrak{M}$ specifically as one associated to a hyperkähler structure (on a product of $K3$ surfaces). For our purposes we need only this “restricted” definition.

Remark 4.11. Again as in Remark 4.5, if $\text{Pic}(S_1) = \text{Pic}(S_2) = \langle 0 \rangle$ for $K3$ surfaces (S_1, I_1) and (S_2, I_2) then $\text{Pic}(S_1 \times S_2) = \langle 0 \rangle$ and any twistor line $Q_{\phi, V}$ is associated

to a hyperkähler structure on $S_1 \times S_2$ determined by (I_1, I_2) and a Kähler class $h + \psi(h) \in H^{1,1}(S_1 \times S_2, \mathbb{R})$. Thus there exists a lift

$$\mathbb{P}_{\phi, V}$$

of $Q_{\phi, V}$ to $\mathfrak{M} \times \mathfrak{M}$.

4.2. Definition of the moduli spaces. Here we introduce two notions of a moduli space. Both of the moduli spaces we want to define involve the moduli space of marked $K3$ surfaces. Moreover, we need the markings in our construction to be *signed*, a notion that we will define later in this subsection. For the $K3$ lattice Λ consider the subset C_Λ ,

$$C_\Lambda = \{\alpha \in \Lambda_{\mathbb{R}} | (\alpha, \alpha) > 0\}.$$

It is an open subset of $\Lambda_{\mathbb{R}}$ and as the signature of Λ is $(3, 19)$, C_Λ retracts onto $V \setminus \{0\}$ for every positive 3-dimensional subspace V of $\Lambda_{\mathbb{R}}$. Hence $H^2(C_\Lambda, \mathbb{Z}) \cong H^2(V \setminus \{0\}, \mathbb{Z})$ is cyclic. Now, an orientation of C_Λ is a choice of a generator in $H^2(C_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$. This choice does not depend on the choice of the positive 3-subspace V . For more details see [21, Section 4].

For Λ considered as the cohomology lattice $H^2(S, \mathbb{Z})$ of a $K3$ surface S we will denote C_Λ by C_S , so that

$$C_S \cap H^{1,1}(S, \mathbb{R}) = \{\alpha \in H^{1,1}(S, \mathbb{R}) | (\alpha, \alpha) > 0\}.$$

The latter set has two connected components. Denote by

$$(7) \quad C_S^+$$

the connected component containing the Kähler cone of S . By definition this is the *positive cone of the $K3$ surface S* . Let us denote the Kähler cone of S as K_S .

Let Λ_1, Λ_2 be $K3$ lattices. Let us fix orientations of C_{Λ_1} and C_{Λ_2} by which we mean generators of the cyclic groups $H^2(C_{\Lambda_1}, \mathbb{Z}), H^2(C_{\Lambda_2}, \mathbb{Z})$.

Definition 4.12. An isometry $\eta : \Lambda_1 \rightarrow \Lambda_2$ is called *signed* if the map $\eta : \Lambda_{1\mathbb{R}} \rightarrow \Lambda_{2\mathbb{R}}$ induces an orientation preserving map from $H^2(C_{\Lambda_1}, \mathbb{Z})$ to $H^2(C_{\Lambda_2}, \mathbb{Z})$.

Definition 4.13. A Hodge isometry $\varphi : H^2(S_1, \mathbb{R}) \rightarrow H^2(S_2, \mathbb{R})$ is called *signed* if it maps $C_{S_1}^+$ to $C_{S_2}^+$. It is called *non-signed* if it is not signed.

Note that for any non-signed isometry φ the isometry $-\varphi$ is signed. Here we note that there is a naturally arising orientation of the positive cone C_{S_i} in

$$H^2(S_i, \mathbb{R}) = ((H^{2,0}(S_i, \mathbb{C}) \oplus H^{0,2}(S_i, \mathbb{C})) \cap H^2(S_i, \mathbb{R})) \oplus H^{1,1}(S_i, \mathbb{R})$$

determined by $C_{S_i}^+$ in $H^{1,1}(S_i, \mathbb{R})$: any choice of $\alpha \in C_{S_i}^+$ gives a positive definite 3-space V with an orientation determined by basis $(\alpha, \operatorname{Re} \sigma, \operatorname{Im} \sigma)$. An orientation of V is a choice of a generator in $H^2(V \setminus \{0\}, \mathbb{Z})$. All this implies that for signed markings $\eta_i : H^2(S_i, \mathbb{Z}) \rightarrow \Lambda, i = 1, 2$, and a signed isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ such that $\psi = \eta_2^{-1} \phi \eta_1 : H^2(S_1, \mathbb{R}) \rightarrow H^2(S_2, \mathbb{R})$ is a Hodge isometry, ψ is actually a signed Hodge isometry.

4.2.1. *The cohomological moduli space.* Now we want to introduce the *cohomological moduli space*. Fix an orientation of the positive cone C_Λ of the $K3$ lattice Λ and a signed isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$.

Definition 4.14. The *cohomological moduli space* associated to a signed isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ is a topological space defined as the set \mathcal{M}_ϕ consisting of all quintuples $((S_1, \eta_1), (S_2, \eta_2), \psi)$ where $(S_1, \eta_1), (S_2, \eta_2)$ are marked $K3$ surfaces and ψ is a signed Hodge isometry,

$$\psi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q}),$$

which satisfies

- $\psi = \eta_2^{-1} \circ \phi \circ \eta_1$;
- $\psi(K_{S_1}) \cap K_{S_2} \neq \emptyset$.

It is clear that \mathcal{M}_ϕ can be considered as a locus in the self-product $\mathfrak{M} \times \mathfrak{M}$ of the moduli space \mathfrak{M} of marked $K3$ surfaces. Actually, this is going to be the locus mentioned in the Introduction. We introduce the formal “Hodge isometry coordinate” ψ considering now quintuples rather than four-tuples for further convenience.

Theorem 4.15. \mathcal{M}_ϕ is a complex-analytic non-Hausdorff manifold.

Proof. To prove that \mathcal{M}_ϕ is a complex-analytic manifold it is sufficient to show that \mathcal{M}_ϕ is an open subset of the complex-analytic manifold $(p, p)^{-1}(\Omega_\phi) \subset \mathfrak{M} \times \mathfrak{M}$ where $(p, p) : \mathfrak{M} \times \mathfrak{M} \rightarrow \Omega_\Lambda \times \Omega_\Lambda$ is the Cartesian product of the period map p . Indeed, we have that $(p, p)^{-1}(\Omega_\phi)$ is a complex-analytic submanifold in $\mathfrak{M} \times \mathfrak{M}$ since the period map $p : \mathfrak{M} \rightarrow \Omega_\Lambda$ is a local analytic isomorphism and Ω_ϕ is a complex-analytic submanifold in $\Omega_\Lambda \times \Omega_\Lambda$.

A point $((S_1, \eta_1), (S_2, \eta_2)) \in (p, p)^{-1}(\Omega_\phi)$ belongs to \mathcal{M}_ϕ precisely when

$$\eta_2^{-1} \circ \phi \circ \eta_1(K_{S_1}) \cap K_{S_2} \neq \emptyset.$$

The fact that this is an open condition is proved in [3, Chapter VIII, Proposition 9.4]. \square

Remark 4.16. There is an obvious forgetful map

$$\pi_\phi : \mathcal{M}_\phi \rightarrow \Omega_\phi,$$

$$((S_1, \eta_1), (S_2, \eta_2), \psi) \mapsto ([\eta_1(\sigma_{S_1})], [\eta_2(\sigma_{S_2})]).$$

Here in the definition of π_ϕ we certainly use that $\psi(H^{2,0}(S_1, \mathbb{C})) = H^{2,0}(S_2, \mathbb{C})$ and $\eta_2\psi = \phi\eta_1$.

Definition 4.17. A *twistor path* in \mathcal{M}_ϕ consists of a finite ordered sequence $\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_k$ of twistor lines in \mathcal{M}_ϕ , such that every pair of consecutive lines intersect, together with a choice of a point of intersection $p_i \in \mathbb{P}_i \cap \mathbb{P}_{i+1}$.

A twistor path is *generic* if each of the chosen points of intersection of consecutive twistor lines corresponds to a pair of $K3$ surfaces with trivial Picard groups.

We know from [11] that every two periods $l_1, l_2 \in \Omega_\Lambda$ can be joined by a generic twistor path in Ω_Λ . Now taking the graph of this twistor path under $\tilde{\phi}$ in Ω_ϕ we get a generic twistor path joining periods $(l_1, \tilde{\phi}(l_1))$ and $(l_2, \tilde{\phi}(l_2))$ in Ω_ϕ . Indeed, such graph is a generic twistor path in \mathcal{M}_ϕ because 1) products of $K3$ ’s corresponding to

points $(q_i, \tilde{\phi}(q_i))$ (as in Definition 4.7) have trivial Picard groups (as Hodge isometric K3 surfaces have equal Picard numbers), and 2), as ϕ is signed, each of the lines of the graph is associated to a hyperkähler structure by Remark 4.11 and thus satisfies the conditions of Definition 4.10.

The following lemma will be used in the proof of Proposition 4.20.

Lemma 4.18. *Let $x = ((X, \eta_X), (Y, \eta_Y), \psi)$ be a point in \mathcal{M}_ϕ , $\alpha \in K_X, \psi(\alpha) \in K_Y$. Then the twistor line $\mathbb{P}_{\psi, \alpha} \subset \mathfrak{M} \times \mathfrak{M}$ lies in \mathcal{M}_ϕ .*

Proof. The cohomology class α determines the line $Q = Q_{\psi, \alpha} \subset \Omega_\phi$. Let λ be a point in $Q \cong S^2$. For its lift $x_\lambda = ((X_\lambda, \eta_{X_\lambda}), (Y_\lambda, \eta_{Y_\lambda}), \psi_\lambda) \in \mathbb{P}_{\psi, \alpha}$ there is a distinguished Kähler class $\omega_\lambda \in K_X$ which was introduced earlier. In order to prove that x_λ belongs to \mathcal{M}_ϕ for every $\lambda \in Q$ it is sufficient to check that $\psi_\lambda(\omega_\lambda) \in K_{Y_\lambda}$ for all $\lambda \in S^2$.

As ϕ and the markings are signed, the basis

$$(\phi(\eta_{X_\lambda}(\omega_\lambda)), \operatorname{Re} \phi(\eta_{X_\lambda}(\sigma_{X_\lambda})), \operatorname{Im} \phi(\eta_{X_\lambda}(\sigma_{X_\lambda})))$$

of the positive definite subspace

$$\langle \phi(\eta_X(\alpha)), \operatorname{Re} \phi(\eta_X(\sigma_X)), \operatorname{Im} \phi(\eta_X(\sigma_X)) \rangle \subset \Lambda_\mathbb{R}$$

is positively oriented and so is the basis

$$(\eta_{Y_\lambda}^{-1} \circ \phi \circ \eta_{X_\lambda}(\omega_\lambda), \operatorname{Re} \eta_{Y_\lambda}^{-1} \circ \phi \circ \eta_{X_\lambda}(\sigma_{X_\lambda}), \operatorname{Im} \eta_{Y_\lambda}^{-1} \circ \phi \circ \eta_{X_\lambda}(\sigma_{X_\lambda})).$$

By Remark 4.4 the class $\psi_{X_\lambda}(\omega_\lambda) = \eta_{Y_\lambda}^{-1}(\phi(\eta_{X_\lambda}(\omega_\lambda)))$ must be Kähler. Now we see that the whole twistor line $\mathbb{P}_{\psi, \alpha}$ lies in \mathcal{M}_ϕ . \square

In the proofs of Proposition 4.20 and Proposition 6.1 below we will need to know how to construct a generic twistor line in $\mathfrak{M} \times \mathfrak{M}$ passing through a given point $x \in \mathcal{M}_\phi$.

Lemma 4.19. *Given $x = ((X, \eta_X), (Y, \eta_Y), \psi) \in \mathcal{M}_\phi$ we can find a generic twistor line $\mathbb{P}_{\psi, h} \subset \mathfrak{M} \times \mathfrak{M}$ for some $h \in K_X \cap \psi^{-1}(K_Y)$. Moreover, in the case of X with cyclic Picard group, $\operatorname{Pic}(X) \cong \mathbb{Z}h, h \in \operatorname{Pic}(X) \cap K_X \cap \psi^{-1}(K_Y)$, the twistor line $\mathbb{P}_{\psi, h}$ is generic.*

Proof. If $\operatorname{Pic}(X \times Y)$ is trivial then the statement of the lemma is tautologically true, any $h \in K_X$ works. Let us assume now that $\operatorname{Pic}(X) \neq \{0\}$. We are going to find $h \in K_X \cap \psi^{-1}(K_Y)$ such that the twistor line $Q_{\psi, h} \subset \Omega_\phi$ contains period of some $X' \times Y'$ with $\operatorname{Pic}(X')$ trivial. Then the corresponding via ϕ surface Y' will also have trivial Picard group. So we will get $\operatorname{Pic}(X' \times Y') = 0$. Thus, the problem is reduced to finding h as above such that the twistor line

$$Q_{X, h} = \mathbb{P}\eta_X(\langle h, \operatorname{Re} \sigma_X, \operatorname{Im} \sigma_X \rangle)_\mathbb{C} \cap Q \subset \Omega_\Lambda,$$

contains period $(t, \tilde{\phi}(t))$ of a K3 surface with trivial Picard group.

Denote the subspace $\langle h, \operatorname{Re} \sigma_X, \operatorname{Im} \sigma_X \rangle$ for $h \in \Lambda_\mathbb{R}$ by W_h . In order for $Q_{X, h}$ to contain a period corresponding to a surface with trivial Picard group we need

$$(8) \quad \eta_X(W_h) \not\subset \bigcup_{0 \neq \lambda \in \Lambda} \lambda^\perp,$$

here the orthogonal complements λ^\perp are taken in $\Lambda_\mathbb{R}$. By the way, notice that $\eta_X(W_h) \subset \bigcup_{0 \neq \lambda \in \Lambda} \lambda^\perp$ is equivalent to existence of nonzero $\lambda \in \Lambda$ such that $\eta_X(W_h) \subset$

λ^\perp . Now it is clear that the condition $W_h \perp \eta_X^{-1}(\lambda)$ for every $h \in K_X \cap \psi^{-1}(K_Y)$ means that $H^{2,0}(X, \mathbb{R}) \oplus H^{0,2}(X, \mathbb{R}) \perp \eta_X^{-1}(\lambda)$ and $H^{1,1}(X, \mathbb{R}) \perp \eta_X^{-1}(\lambda)$ which contradicts the nondegeneracy of the form $q(\cdot, \cdot)$. Moreover, we get the same contradiction if we assume $W_h \perp \eta_X^{-1}(\lambda)$ for every $h \in \text{Pic}(X) \cap K_X \cap \psi^{-1}(K_Y)$. So there exists an h such that W_h determines a twistor line containing a point t with X_t having trivial Picard group. Fix this h .

As ψ is a rational isometry, the Picard group of $Y_{\tilde{\phi}(t)}$ is also trivial. Thus, we can find a period $(t, \phi(t)) \in Q_{\psi, h} \subset \Omega_\phi$ determining surfaces X', Y' with trivial Picard groups together with their markings and a rational isometry determining a point $((X', \eta_{X'}), (Y', \eta_{Y'}), \psi') \in \mathcal{M}_\phi$ with $\text{Pic}(X' \times Y') = \langle 0 \rangle$. It is also clear that if $\text{Pic}(X) \cong \mathbb{Z}h$ for $h \in K_X \cap \psi^{-1}(K_Y)$, then condition (8) is satisfied and thus $\mathbb{P}_{\psi, h}$ is generic. \square

Proposition 4.20. *The moduli space \mathcal{M}_ϕ consists of 2 connected components \mathcal{M}_ϕ^+ and \mathcal{M}_ϕ^- . The component \mathcal{M}_ϕ^+ consists of quintuples $((S, \eta), (S', \eta'), \psi)$ where both η, η' are signed, and \mathcal{M}_ϕ^- consists of such quintuples where both η, η' are non-signed isometries. Moreover, for any two points in the same connected component there exists a twistor path in the corresponding connected component consisting of lines of the form (6) and connecting these points.*

Proof. First of all, set-theoretically $\mathcal{M}_\phi = \mathcal{M}_\phi^+ \sqcup \mathcal{M}_\phi^-$. Now let us prove connectedness of each of the two sets in the decomposition. Let us prove that \mathcal{M}_ϕ^+ is connected.

We are going to show that any two points of \mathcal{M}_ϕ^+ can be joined by a twistor path. First, pick up a point $y = ((S, \eta_S), (T, \eta_T)) \in \mathfrak{M} \times \mathfrak{M}$ with signed markings η_S, η_T and periods $\eta_S(H^{2,0}(S, \mathbb{C})) = l$ and $\eta_T(H^{2,0}(T, \mathbb{C})) = \tilde{\phi}(l)$ in Ω_Λ , such that the Picard groups $\text{Pic}(S)$ and $\text{Pic}(T)$ are trivial. We can certainly find such (S, η_S) and (T, η_T) with periods l and $\tilde{\phi}(l)$ respectively, because of surjectivity of the period map $\pi : \mathfrak{M} \rightarrow \Omega_\Lambda$ and the fact that periods corresponding to surfaces with nontrivial Picard group are contained in a countable union of hyperplanes $\bigcup_{\lambda \in \Lambda} \mathbb{P}\lambda_\mathbb{C}^\perp \subset \mathbb{P}\Lambda_\mathbb{C}$.

Once we choose l so as to have $\text{Pic}(S) = \langle 0 \rangle$, the group $\text{Pic}(T)$ is also trivial because $\psi = \eta_T^{-1} \phi \eta_S$ is a rational Hodge isometry, hence the Picard groups of S and T are isomorphic.

Moreover, $\psi = \eta_T^{-1} \phi \eta_S$ is a signed Hodge isometry, so it takes C_S^+ to C_T^+ . As for K3 surfaces with trivial Picard group the Kähler cone equals to the positive cone (see Remark 4.5), we conclude that point y indeed belongs to \mathcal{M}_ϕ^+ . Such a choice of y means that (S, η_S) and (T, η_T) are the only points in the fibers of the period map $\pi : \mathfrak{M} \rightarrow \Omega_\Lambda$ over l and $\tilde{\phi}(l)$. Indeed, let, for example, $(X, \eta_X), (Y, \eta_Y)$ be two distinct points such that all markings are signed and

$$\pi(X, \eta_X) = \pi(Y, \eta_Y) = l.$$

Now $g = \eta_Y^{-1} \eta_X : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$ is a signed Hodge isometry. So, by the strong Torelli theorem for K3 surfaces, see [27], there is an isomorphism $f : X \rightarrow Y$ and $w \in W(Y)$, the Weil group of Y , generated by reflections with respect to hyperplanes in $H^2(Y, \mathbb{Q})$ orthogonal to divisors corresponding to (-2) -curves on Y , such that $g = w \circ f^*$. But $W(Y)$ is trivial by the choice of Y . So $g = f^*$ and we have that $\eta_Y \circ f^* = \eta_X$, so that f^* induces an isomorphism of marked pairs $(X, \eta_X), (Y, \eta_Y)$ which

means that they represent the same point of moduli space of marked $K3$ surfaces. The same argument applies to the fiber of π over $\tilde{\phi}(l)$. For such choice of l and $\tilde{\phi}(l)$ the point y itself is a unique point in the fiber of π_ϕ over $(l, \tilde{\phi}(l))$.

Now choose an arbitrary point

$$x = ((X, \eta_X), (Y, \eta_Y), \psi) \in \mathcal{M}_\phi^+.$$

We want to connect this arbitrary point to the point y by a twistor path in \mathcal{M}_ϕ^+ consisting of lines of the form (6) which is constructed as a connected lift of a twistor path joining the periods $\pi_\phi(x)$ and $\pi_\phi(y)$ in Ω_ϕ .

So, first, choose a generic path of twistor lines Q_1, \dots, Q_k in Ω_ϕ joining the pairs of periods $([\eta_X(\sigma_X)], [\eta_Y(\sigma_Y)]), ([\eta_S(\sigma_S)], [\eta_T(\sigma_T)]) \in \Omega_\phi$ which exists by discussion after Definition 4.17. Consider points $q_i \in Q_i \cap Q_{i+1}, i = 1, \dots, k-1$, as in Definition 4.17, $q_0 := \pi_\phi(x), q_k := \pi_\phi(y)$. By the same discussion after Definition 4.17 we see that Q_2, \dots, Q_k admit lifts to $\mathfrak{M} \times \mathfrak{M}$ via lines of the form (6) and by Remark 4.11 the union of their lifts is connected. Actually, line Q_1 admits a lift to \mathcal{M}_ϕ through the lift of q_1 as well and if it contains x we are done. However this may not be the case in general, so we certainly need Q_1 to have a lift through x . Existence of Q_1 with this property is provided by Lemma 4.19. A point $z \in Q_1$ with trivial Picard group can be taken to be q_1 , and then Q_2, \dots, Q_k may be taken to be any generic twistor path joining z to y . Then the whole path of twistor lines admits a lift of the required form to $\mathfrak{M} \times \mathfrak{M}$ through the point $x = ((X, \eta_X), (Y, \eta_Y), \psi)$.

Moreover, as η_X and η_Y were chosen signed, all the markings $\eta_{X_\lambda}, \eta_{Y_\lambda}$, λ – the parameter of a twistor line in the path, will also be signed. Now using our special choice of y we see that we thus get a path consisting of lines of the form

$$\lambda \mapsto ((X_\lambda, \eta_{X_\lambda}), (Y_\lambda, \eta_{Y_\lambda}))$$

in the self-product $\mathfrak{M} \times \mathfrak{M}$ joining x and y and being a lift of the corresponding twistor path in Ω_ϕ with respect to the obvious projection. Moreover, by construction $\psi_\lambda = \eta_{Y_\lambda}^{-1} \phi \eta_{X_\lambda}$ is a rational Hodge isometry. Each of the lines of this path lies in \mathcal{M}_ϕ^+ by Lemma 4.18. Now we see that the whole twistor path lifts to \mathcal{M}_ϕ^+ and, moreover, it connects x to y because it contains a lift of the $\pi_\phi(y)$ with respect to π_ϕ and y was chosen so that it is the only point in the fiber of π_ϕ over its period.

So we know that we can join any point $x \in \mathcal{M}_\phi^+$ to $y \in \mathcal{M}_\phi^+$. Finally, we see that we can join any $x, z \in \mathcal{M}_\phi^+$ by lifts of twistor paths.

The proof of connectedness of \mathcal{M}_ϕ^- goes analogously. This completes the proof of Proposition 4.20. \square

4.2.2. The sheaf moduli space.

Definition 4.21. The *sheaf moduli space* associated to a signed isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ is the set $\overline{\mathcal{M}}_\phi$ of quintuples $((S_1, \eta_1), (S_2, \eta_2), \mathcal{E})$, where $(S_i, \eta_i), i = 1, 2$, are marked $K3$ surfaces and \mathcal{E} is a twisted sheaf on $S_1 \times S_2$ satisfying the following conditions

- \mathcal{E} is a locally free sheaf;
- the cohomology class $-\kappa(\mathcal{E}^*)\sqrt{td_{S_1 \times S_2}}$ determines a Hodge isometry $\psi_{\mathcal{E}} : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q})$, such that $\psi_{\mathcal{E}}(K_{S_1}) \cap K_{S_2} \neq \emptyset$;
- $\psi_{\mathcal{E}} = \eta_2^{-1} \phi \eta_1$.

Note that here we define $\psi_{\mathcal{E}}$ as the isometry determined by $-\kappa(\mathcal{E}^*)\sqrt{td_{S_1 \times S_2}}$ instead of $\kappa(\mathcal{E}^*)\sqrt{td_{S_1 \times S_2}}$ as in Section 2.1. The reason for introducing minus is that having in mind our Important example of Subsection 3.3 we want our Hodge isometry to satisfy the second requirement of definition of $\overline{\mathcal{M}}_{\phi}$, that is, the Kähler cone condition. That $-\kappa(\mathcal{E}^*)\sqrt{td_{S_1 \times S_2}}$ satisfies this condition will essentially be shown in the proof of Proposition 4.24. Actually, $\overline{\mathcal{M}}_{\phi}$ has a structure of a complex-analytic non-Hausdorff manifold, which will be proved in a forthcoming paper. For our purposes here it is sufficient to consider the sheaf moduli space set-theoretically.

Notation 4.22. The forgetful map

$$\overline{p} : \overline{\mathcal{M}}_{\phi} \rightarrow \mathcal{M}_{\phi},$$

is the map defined by

$$((S_1, \eta_1), (S_2, \eta_2), \mathcal{E}) \mapsto ((S_1, \eta_1), (S_2, \eta_2), \psi_{\mathcal{E}}).$$

Remark 4.23. Let m be a point $((X, \eta_X), (Y, \eta_Y), \psi)$ of \mathcal{M}_{ϕ} , X and Y are $K3$ surfaces, such that there exists a point $((X, \eta_X), (Y, \eta_Y), \mathcal{E})$ in $\overline{\mathcal{M}}_{\phi}$ so that $\psi = \psi_{\mathcal{E}}$. We can simply express this fact by saying that m belongs to the image of \overline{p} . It is now clear that for such points m in the image of \overline{p} the corresponding ψ is represented by a class of analytic type (mentioned in the Introduction, see also Definition 6.2) in $H^*(X \times Y, \mathbb{Q})$. Now let X and Y be algebraic $K3$ surfaces. Then ψ is algebraic. So the fact that rational Hodge isometries between algebraic $K3$ surfaces, appearing as a coordinate in such a quintuple in \mathcal{M}_{ϕ} , are algebraic would follow if the map \overline{p} were surjective.

Introduce the following subsets of $\overline{\mathcal{M}}_{\phi}$,

$$\overline{\mathcal{M}}_{\phi}^+ = \overline{p}^{-1}(\mathcal{M}_{\phi}^+),$$

and

$$\overline{\mathcal{M}}_{\phi}^- = \overline{p}^{-1}(\mathcal{M}_{\phi}^-).$$

Now we need examples illustrating the introduced notions and for examples we refer to Subsection 3.3. According to the general example there it appears that the rational Hodge isometry determined by $\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$ for \mathcal{E} a universal sheaf on $S \times M$ are of a very specific type. Via markings of the corresponding surfaces such isometries induce rational isometries $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ of *cyclic type*, a notion that was introduced in the Introduction. Recall the equivalent definition of isometries of cyclic type given in Section 3. We say that a rational isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ is of *n-cyclic type* if $\Lambda/(\Lambda \cap \phi^{-1}(\Lambda)) \cong \mathbb{Z}/n\mathbb{Z}$. Isometries of cyclic type are, in some sense, the simplest isometries of $\Lambda_{\mathbb{Q}}$. Moreover, for every such isometry ϕ we can find algebraic $K3$ surfaces S_1, S_2 and appropriate markings so that ϕ determines a rational Hodge isometry of the cohomology lattices of the surfaces and this Hodge isometry is induced by the κ -class of a twisted sheaf. This is proved in the following proposition.

Proposition 4.24. *For a signed isometry ϕ of cyclic type the moduli space $\overline{\mathcal{M}}_{\phi}$ is not empty.*

Proof. Let r be the natural number such that ϕ is of r -cyclic type. The proof of the proposition is essentially given by Conclusion 3.10, which provides existence of

marked $K3$ surfaces (S, η_S) , (M, η_M) and a locally free sheaf \mathcal{E} over $S \times M$, such that $\kappa(\mathcal{E})\sqrt{td_{S \times M}}$ induces an r -cyclic type isometry $\psi : H^2(S, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$. The only thing that needs to be verified is that $\psi_{\mathcal{E}} = -\psi$ satisfies the Kähler cone condition in the definition of $\overline{\mathcal{M}}_{\phi}$.

As $\hat{h}^2 = h^2$ and $\psi_{\mathcal{E}}$ restricts to an isomorphism between $\mathbb{Q}h$ and $\mathbb{Q}\hat{h}$, we certainly have that $\psi_{\mathcal{E}}(h) = \pm\hat{h}$. Let us show that $\psi_{\mathcal{E}}(h) = \hat{h}$. The isometry $\psi_{\mathcal{E}}$ is induced by the $H^4(S \times M, \mathbb{Q})$ -component of $-\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$. We have

$$\begin{aligned} \kappa(\mathcal{E}^*)\sqrt{td_{S \times M}} &= ch(\mathcal{E}^*)e^{-\frac{c_1(\mathcal{E}^*)}{r}}\sqrt{td_{S \times M}} = \\ &= \left(r + c_1(\mathcal{E}^*) + \frac{c_1(\mathcal{E}^*)^2 - 2c_2(\mathcal{E}^*)}{2} + \dots \right) \cdot \left(1 - \frac{c_1(\mathcal{E}^*)}{r} + \frac{c_1(\mathcal{E}^*)^2}{2r^2} - \dots \right) \times \\ &\quad \times (1 + \pi_S^* \mathbf{1}_S + \pi_M^* \mathbf{1}_M + \pi_S^* \mathbf{1}_S \cdot \pi_M^* \mathbf{1}_M) = \\ &= r + \left(\frac{(r-1)c_1(\mathcal{E}^*)^2}{2r} - c_2(\mathcal{E}^*) + \pi_S^* \mathbf{1}_S + \pi_M^* \mathbf{1}_M \right) + \dots \end{aligned}$$

As $\pi_S^* \mathbf{1}_S$ and $\pi_M^* \mathbf{1}_M$ induce zero maps $H^2(S, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$ we need to consider only

$$\frac{(r-1)c_1(\mathcal{E}^*)^2}{2r} - c_2(\mathcal{E}^*).$$

In

$$c_1(\mathcal{E}^*)^2 = (-\pi_S^* h - j\pi_M^* \hat{h})^2 = \pi_S^*(2rs\mathbf{1}_S) + \pi_M^*(2rsj^2\mathbf{1}_M) + 2\pi_S^* h \cdot \pi_M^* j\hat{h}$$

the only component inducing nonzero map is $2\pi_S^* h \cdot \pi_M^* j\hat{h}$. Denote the image of h in $H^2(M, \mathbb{Z})$ under $c_2(\mathcal{E}^*) = c_2(\mathcal{E})$ by ψ , following the notations of [23]. As Mukai showed in [23, Proposition 1.1], $\psi = [1 + (2(r-1)sj)]\hat{h}$. The image of h under $2\pi_S^* h \cdot \pi_M^* j\hat{h}$ is $4rsj\hat{h}$. Finally we get that $\psi_{\mathcal{E}}$ maps h to

$$\begin{aligned} \pi_{M*} \left(\pi_S^* h \cdot \left(c_2(\mathcal{E}^*) - \frac{(r-1)c_1(\mathcal{E}^*)^2}{2r} \right) \right) &= \psi - \frac{4(r-1)rsj\hat{h}}{2r} = \\ &= \psi - 2(r-1)sj\hat{h} = \hat{h}. \end{aligned}$$

So the Kähler cone condition for $\psi_{\mathcal{E}}$ is verified and the proof of the proposition is complete. \square

5. HYPERHOLOMORPHIC SHEAVES

Here we formulate some results on extension of sheaves over twistor families that we will need in the next section. The original definition of a hyperholomorphic vector bundle can be found in [30, Definition 3.11]. Here we give a more convenient for us definition which is equivalent to the original one.

Assume N is a hyperkähler manifold, I is a fixed complex structure on N and H is a Kähler class on N . There exists a hyperkähler metric associated to a $(1,1)$ -form representing the cohomology class H . Then we have a sphere of complex structures on N , a twistor family $\mathcal{N} \rightarrow S^2$, and a Kähler class ω_{λ} on the fiber N_{λ} for each $\lambda \in S^2$, such that $N_I = N$ and $\omega_I = H$.

Remark 5.1. By definition, the class ω_{λ} (considered up to multiplying by a positive scalar) is the Kähler class on (N, λ) .

Fix these H, S^2 and \mathcal{N} . A vector bundle F on (N, I) is called *H-hyperholomorphic* if it can be extended to a vector bundle over \mathcal{N} . The Kähler class H defines the notion of H -slope-stability and H -slope-polystability of vector bundles on N . Recall that a vector bundle on N is called *H-slope-polystable* if it is isomorphic to a direct sum of H -slope-stable bundles with equal slopes, see an equivalent definition in [13, Definition 1.5.4].

Let us formulate a theorem proved by M. Verbitsky (see [30, Theorems 3.17, 3.19], keeping in mind Remark 5.1).

Theorem 5.2. *Let F be an H -slope-polystable vector bundle over (N, I) . If the Hodge types of $c_1(F)$ and $c_2(F)$ are preserved under the deformation identified with the sphere S^2 of complex structures on N , then the bundle F extends over \mathcal{N} . Furthermore, the extended vector bundle \mathcal{F} over \mathcal{N} restricts to the fiber N_λ as an ω_λ -slope-polystable bundle, for each $\lambda \in S^2$.*

Remark 5.3. Let $\pi : \mathcal{X} \rightarrow B$ be a smooth and proper holomorphic map of complex manifolds with a continuous trivialization $\eta : R^k\pi_*\mathbb{Z} \rightarrow (\Lambda)_B$ of the local system, where $(\Lambda)_B$ is the trivial local system with fiber Λ . We get an induced flat trivialization of the local system $R^k\pi_*\mathbb{C}$, which we denote by η as well. Let 0 be a point of B . Denote by X_b the fiber of π over $b \in B$. The flat deformation of a class $\alpha \in H^k(X_0, \mathbb{C})$ in the local system $R^k\pi_*\mathbb{C}$ associated to the family is given by the section $b \mapsto \eta_b^{-1}\eta_0(\alpha)$ of the local system $R^k\pi_*\mathbb{C}$.

Let \mathcal{F} be a vector bundle over \mathcal{X} . Each Chern class of \mathcal{F} defines a flat section of the local system. Deformations of the Chern classes of the vector bundle $F = \mathcal{F}_0$ on X_0 agree with taking Chern classes of the extension, that is, $\eta_b^{-1}\eta_0(c_i(\mathcal{F}_0)) = c_i(\mathcal{F}_b) \in H^*(X_b, \mathbb{C})$.

Remark 5.4. Theorem 5.2 addresses existence of extensions of sheaves on N over a family over the sphere S^2 of complex structures on N . In Section 6 we will need to extend sheaves over families of products of $K3$'s over twistor lines in Ω_ϕ . In order to use Theorem 5.2 we need to associate to a twistor line $Q_{\psi, h}$ an appropriate sphere of complex structures S^2 on N . While such association trivially exists in the case when N is a $K3$ surface, it becomes nontrivial when N is a product of $K3$'s. Lemma 4.9 provides a way to associate a hyperkähler structure to a twistor line (we may take any representative of the special class in \mathcal{T}/D , whose existence is stated in the Lemma).

We are now going to make some preparations needed for the proof of Proposition 6.1 and related to the problem of extending sheaves over twistor families. Let \mathcal{N}, \mathcal{F} and H be as in Theorem 5.2 for the rest of this section. Consider now a hyperkähler manifold $N = S \times M$ where (S, η_S) and (M, η_M) , $M = M_h(v)$, are marked $K3$ surfaces and \mathcal{E} is a universal untwisted sheaf on $S \times M$. We will specify the precise choice of S, M and \mathcal{E} that is used in the proof of Proposition 6.1 in the formulation of Lemma 5.7 below. Ideally we would like to be able to extend \mathcal{E} over a twistor family through $((S, \eta_S), (M, \eta_M), \psi_\mathcal{E})$ determined by $H = (h, \psi_\mathcal{E}(h))$ which corresponds to a twistor line in \mathcal{M}_ϕ where

$$\phi = \eta_M \circ \psi_\mathcal{E} \circ \eta_S^{-1} : \Lambda_\mathbb{Q} \rightarrow \Lambda_\mathbb{Q},$$

$$\phi \in \Lambda_\mathbb{Q}^* \otimes \Lambda_\mathbb{Q}.$$

The Hodge types of $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ need not be preserved under such twistor deformation and thus we cannot, in general, hope for extending \mathcal{E} as an untwisted sheaf. But the following lemma gives the first step towards understanding what should be our extendable substitute for \mathcal{E} .

Lemma 5.5. *Let $((X, \eta_X), (Y, \eta_Y), \psi)$ be a point in \mathcal{M}_ϕ . The Hodge type of the cohomology class given by the Hodge isometry coordinate*

$$\psi \in H^2(X, \mathbb{Q})^* \otimes H^2(Y, \mathbb{Q}) \subset H^4(X \times Y, \mathbb{Q})$$

is preserved along any twistor line in \mathcal{M}_ϕ (see notation (6)) through $((X, \eta_X), (Y, \eta_Y), \psi)$.

Proof. For any $t \in Q_{\psi, h}$, h a Kähler class in $H^{1,1}(X, \mathbb{R})$, $\psi(h)$ a Kähler class in $H^{1,1}(Y, \mathbb{R})$, we have that, first,

$$(\eta_{X_t}^*{}^{-1} \otimes \eta_{Y_t}) : H^2(X_t, \mathbb{Z})^* \otimes H^2(Y_t, \mathbb{Z}) \rightarrow \Lambda^* \otimes \Lambda,$$

$$\alpha \mapsto \eta_{Y_t} \circ \alpha \circ \eta_{X_t}^{-1}$$

is a marking for $Z_t = X_t \times Y_t$ defined by the corresponding markings of the factors (see our convention on markings of products $X_t \times Y_t$ in Remark 4.1, we identify η_{X_t} with $\eta_{X_t}^*$ via identification of the dual lattices $H^2(X_t, \mathbb{Z})^*, \Lambda^*$ with $H^2(X_t, \mathbb{Z}), \Lambda$ using the corresponding unimodular bilinear pairings). Here by X_t and Y_t we denote the differentiable manifolds X and Y with complex structures depending on $t \in Q_{\psi, h}$. And, second, by the definition of a twistor line in \mathcal{M}_ϕ we have that the Hodge isometry coordinate of the point in \mathcal{M}_ϕ corresponding to t satisfies,

$$\psi_t = \eta_{Y_t}^{-1} \circ \phi \circ \eta_{X_t} = \eta_{Y_t}^{-1} \circ (\eta_Y \circ \psi_\mathcal{E} \circ \eta_X^{-1}) \circ \eta_{X_t} = \eta_{X_t \times Y_t}^{-1} \circ \eta_{X \times Y}(\psi_\mathcal{E}),$$

which shows that ψ_t is the deformation of $\psi_\mathcal{E}$ according to Remark 5.3. The isometry $\psi_t : H^2(X_t, \mathbb{Q}) \rightarrow H^2(Y_t, \mathbb{Q})$ is certainly a Hodge isometry, which follows from the equality $\psi_t(H^{2,0}(X_t, \mathbb{C})) = H^{2,0}(Y_t, \mathbb{C})$. But this condition precisely means that ψ_t , considered as a class in $H^4(X_t \times Y_t, \mathbb{C})$ is of type $(2, 2)$. So the Hodge type of ψ is preserved along the twistor line. \square

Let us apply Lemma 5.5 to the point $((S, \eta_S), (M, \eta_M), \psi_\mathcal{E}) \in \mathcal{M}_\phi$. The isometry $\psi_\mathcal{E} : H^2(S, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$ is actually induced by the degree 4 component of the class $\kappa(\mathcal{E}^*)\sqrt{td_{S \times M}}$ as we saw in Subsection 3.3. Moreover, as the Hodge types of elements of degree 0 and 4 in $H^*(S, \mathbb{Q})$ and $H^*(M, \mathbb{Q})$ are trivially preserved under any deformations, and thus the Hodge type of Todd classes of S and M is also preserved, the preservation of the Hodge type of $\psi_\mathcal{E}$ by Lemma 5.5 implies preservation of the Hodge type of $\kappa_2(\mathcal{E}^*)$. And now we describe the last step towards finding an appropriate substitute for \mathcal{E} . Consider the sheaf $\mathcal{A} = \text{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ instead of \mathcal{E} . We have

$$c_1(\mathcal{A}) = 0$$

and

$$c_2(\mathcal{A}) = -2r\kappa(\mathcal{E}),$$

where $r = rk(\mathcal{E})$. Let us explain the last equality. Recall that

$$\kappa(\mathcal{A}) = ch(\mathcal{A})exp(-c_1(\mathcal{A})/r^2) = ch(\mathcal{A}),$$

see Subsection 2.1. Now, comparing the components of $\kappa(\mathcal{A})$ and $ch(\mathcal{A})$ of degree 4 we get

$$c_2(\mathcal{A}) = -ch_2(\mathcal{A}) = -\kappa_2(\mathcal{A}) = -\kappa_2(\mathcal{E}^* \otimes \mathcal{E}) = -2r\kappa_2(\mathcal{E}).$$

In the last equality we used the multiplicativity of the class κ on tensor products, the fact that κ_1 of any sheaf is zero and $\kappa_{2i}(\mathcal{E}) = \kappa_{2i}(\mathcal{E}^*)$ for any $i \geq 0$. So the class $-2r\kappa_2(\mathcal{E})$ naturally serves as the second Chern class for the sheaf of Azumaya algebras \mathcal{A} . The Hodge type of the class $\kappa(\mathcal{E})$ is preserved under the considered twistor deformations together with the Hodge type of $\kappa(\mathcal{E}^*)$, and thus the Hodge type of $c_2(\mathcal{A})$ is preserved under the considered twistor deformations. Thus the condition of Hodge type invariance in Theorem 5.2 for $c_1(\mathcal{A}) = 0$ and $c_2(\mathcal{A})$ is satisfied.

The only condition that remains to check is that \mathcal{A} is H -slope-polystable. Note that in order to prove that \mathcal{A} is H -slope-polystable it is sufficient to show that \mathcal{E} is H -slope-stable. Indeed, then by [13, Theorem 3.2.11] we would have that $\mathcal{A} = \mathcal{E}^* \otimes \mathcal{E}$ is H -slope-polystable and we may thus set $\mathcal{F} := \mathcal{A}$. Thus by Theorem 5.2 the sheaf \mathcal{A} can be deformed as an untwisted sheaf.

Remark 5.6. More precisely, this twistor deformation preserves the structure of Azumaya algebra, see Section 6 of [19].

As we know from the correspondence between sheaves of Azumaya algebras and twisted sheaves (see, for example, [7]) deforming the sheaf \mathcal{A} as a sheaf of Azumaya algebras is the same as deforming the sheaf \mathcal{E} as a twisted sheaf. So Remark 5.6 tells us that the twistor deformation of $\mathcal{F} = \mathcal{A}$ provided by Theorem 5.2 determines a deformation of \mathcal{E} as a twisted sheaf.

Let us get to the problem of proving the stability of \mathcal{E} . In general the universal sheaf \mathcal{E} need not be slope-stable with respect to an arbitrary divisor H but there exists a particular choice of S, M, \mathcal{E} and H such that \mathcal{E} is H -slope-stable.

Lemma 5.7. *Consider K3 surfaces S, M , divisors $h \in \text{Pic}(S)$, $\hat{h} \in \text{Pic}(M)$, and the universal sheaf \mathcal{E} on $S \times M$ as in the proof of Proposition 4.24. Then \mathcal{E} is slope-stable with respect to $\pi_S^*h + \pi_M^*\hat{h} \in \text{Pic}(S \times M)$. Consequently, the Azumaya algebra $\mathcal{A} := \text{End}(\mathcal{E})$ is $(\pi_S^*h + \pi_M^*\hat{h})$ -hyperholomorphic.*

Proof. In order to prove slope-stability of \mathcal{E} with respect to the divisor $\pi_S^*h + \pi_M^*\hat{h}$ in $\text{Pic}(S \times M) \otimes \mathbb{Q}$ we need to use the slope-stability of $\mathcal{E}|_{S \times \{m\}}$, $m \in M$, with respect to the divisor $h \in \text{Pic}(S)$ which follows from the definition of M and \mathcal{E} , and slope-stability of $\mathcal{E}|_{\{s\} \times M}$, $s \in S$, with respect to the ample divisor $\hat{h} \in \text{Pic}(M)$, which follows from Mukai's paper [23, Theorem 1.2]. When we know that both the vertical and the horizontal restrictions of the sheaf \mathcal{E} are slope-stable, it is sufficient to express the slope $\mu_{H+\hat{H}}$ on $S \times M$, where $H = \pi_S^*h$, $\hat{H} = \pi_M^*\hat{h}$, as a linear combination of slopes μ_h and $\mu_{\hat{h}}$ (composed respectively with the morphisms of restriction of sheaves to $S \times \{m\}$ and $\{s\} \times M$) with positive coefficients. Note that due to the fact that $\text{Pic}(S) \cong \text{Pic}(M) \cong \mathbb{Z}$ we will also have stability for any (rational) divisor of the form $aH + b\hat{H}$, $a, b \in \mathbb{Q}^+$, so for any ample divisor from $\text{Pic}(S \times M)$, although we do not need this more general fact for our proof.

Now, by the definition of slope

$$\mu_{H+\hat{H}}(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot (H + \hat{H})^3}{rk(\mathcal{F})}.$$

First of all we know that $c_1(\mathcal{F}) = \alpha + \beta$ where $\alpha = \pi_S^* c_1(\mathcal{F}|_{S \times \{m\}})$, $\beta = \pi_M^* c_1(\mathcal{F}|_{\{s\} \times M})$. Let us look more closely at the triple self-intersection

$$(H + \hat{H})^3 = H^3 + 3H^2 \cdot \hat{H} + 3H \cdot \hat{H}^2 + \hat{H}^3.$$

Obviously $H^3 = \hat{H}^3 = 0$, $H^2 \cdot \hat{H} = 2d(\{s\} \times M) \cdot \hat{H} = 2d(\{s\} \times \hat{h})$ and $H \cdot \hat{H}^2 = H \cdot 2d(S \times \{m\}) = 2d(h \times \{m\})$ where h and \hat{h} in the direct products are considered set-theoretically. We trivially have that $\alpha \cdot (\{s\} \times \hat{h}) = 0$ and $\beta \cdot (h \times \{m\}) = 0$. Now taking the intersection of $(H + \hat{H})^3$ with $c_1(\mathcal{F})$ we see that

$$\mu_{H+\hat{H}}(\mathcal{F}) = 6d(\mu_h(\mathcal{F}|_{S \times \{m\}}) + \mu_{\hat{h}}(\mathcal{F}|_{\{s\} \times M})),$$

where the degree d of h and \hat{h} is positive. The required stability of \mathcal{E} is proved. \square

6. PROOF OF THE MAIN RESULT

The core result of this section is Proposition 6.1, which states that for each of the subsets $\overline{\mathcal{M}}_\phi^\pm \subset \overline{\mathcal{M}}_\phi$, ϕ an isometry of cyclic type, the restriction $\overline{p} : \overline{\mathcal{M}}_\phi^\pm \rightarrow \mathcal{M}_\phi^\pm$ sending (S_1, S_2, \mathcal{E}) to $(S_1, S_2, -\kappa(\mathcal{E}^*) \sqrt{td_{S_1 \times S_2}})$ is surjective. The proof of Proposition 6.1 is given in Subsection 6.1. Subsection 6.2 explains why Theorem 1.1 formulated in the introduction follows from Proposition 6.1.

6.1. Surjectivity of \overline{p} . In this subsection the isometry ϕ is assumed to be of cyclic type. Let \mathcal{M}_ϕ^+ and \mathcal{M}_ϕ^- be the connected components of \mathcal{M}_ϕ introduced in Proposition 4.20 and $\overline{\mathcal{M}}_\phi^\pm$ the sets defined in Subsection 4.2.2. The following proposition is proved using the technique of hyperholomorphic sheaves developed by M. Verbitsky.

Proposition 6.1. *The forgetful map $\overline{p} : \overline{\mathcal{M}}_\phi \rightarrow \mathcal{M}_\phi$ restricts to a surjective map $\overline{p} : \overline{\mathcal{M}}_\phi^\pm \rightarrow \mathcal{M}_\phi^\pm$.*

Proof. The arguments for each of the two choices of the sign \pm are absolutely similar, so we will be proving surjectivity of $\overline{p} : \overline{\mathcal{M}}_\phi^+ \rightarrow \mathcal{M}_\phi^+$. Let us first outline the proof of surjectivity. For an arbitrary point $y = ((S_1, \eta_1), (S_2, \eta_2), \psi_y) \in \mathcal{M}_\phi^+$ we want to find a preimage of this point in $\overline{\mathcal{M}}_\phi^+$ under \overline{p} . First, we pick up a point $\overline{x} \in \overline{\mathcal{M}}_\phi^+$, which exists by Proposition 4.24. Let $x = \overline{p}(\overline{x}) \in \mathcal{M}_\phi^+$. We choose an appropriate connected twistor path $l \subset \mathcal{M}_\phi^+$ consisting of twistor lines $\mathbb{P}_j, j = 1, \dots, k$, joining x and y , $x \in \mathbb{P}_1$, $y \in \mathbb{P}_k$. We then show that there exist locally free twisted sheaves \mathcal{F}_j over complex-analytic families $\mathcal{X}_j = (i|_{\mathbb{P}_j})^* \mathcal{Y}$ of products of marked $K3$'s, for the inclusion $i : l \rightarrow \mathfrak{M} \times \mathfrak{M}$ and the universal family $\mathcal{Y} \rightarrow \mathfrak{M} \times \mathfrak{M}$ so that $\mathcal{X}_k|_y = (S_1 \times S_2, (\eta_1, \eta_2))$. Thus

$$y = \overline{p}(((S_1, \eta_1), (S_2, \eta_2), \mathcal{F}_k|_{\mathcal{X}_k|_y}))$$

and we are done.

Let us now get to the details of the sketched construction. We want and can choose the point \overline{x} to be of the form $((S, \eta_S), (M, \eta_M), \mathcal{E})$ with $M = M_h(v)$ and \mathcal{E} a universal sheaf on $S \times M$. Let us specify the choice of S and \mathcal{E} . Choose S as in the proof of

Conclusion 3.10 to be a $K3$ surface with a cyclic Picard group generated by an ample divisor h , so that S is general in the sense of Mukai, [23]. Then $M = M_h(v)$ has also a cyclic Picard group, as rationally Hodge isometric $K3$ surfaces have the same Picard numbers. Choose \mathcal{E} to be a normalized universal sheaf on $S \times M$ as in the proof of Conclusion 3.10. We choose l to be a generic twistor path in \mathcal{M}_ϕ joining $\bar{p}(\bar{x})$ and y and consisting of lifts \mathbb{P}_j of lines $Q_j \subset \Omega_\phi$ associated to hyperkähler structures. Such a path was shown to exist in the course of the proof of Proposition 4.20, where \mathcal{M}_ϕ^+ was shown to be connected. Recall that a twistor path comes with a choice of points p_j in the intersection of the lines \mathbb{P}_j and \mathbb{P}_{j+1} (see Definition 4.17).

We are going to extend consequently $\mathcal{A} = \mathcal{E}nd(\mathcal{E})$ over the families \mathcal{X}_j . This means that having extended \mathcal{A} over \mathcal{X}_j we get the extension over \mathcal{X}_{j+1} provided that the conditions of Theorem 5.2 are satisfied for the result of extension of \mathcal{A} restricted to $\mathcal{X}_j|_{p_j}$ for $p_j \in \mathbb{P}_j \cap \mathbb{P}_{j+1}$ for all $j = 1, \dots, k-1$, $p_0 := x$.

In order to extend \mathcal{A} from $\mathcal{X}_1|_{p_0} = ((S, \eta_S), (M, \eta_M))$ over the whole \mathcal{X}_1 , we want to choose \mathbb{P}_1 in the path l to be the line determined by $h \in Pic(S)$. Indeed, as $\psi_{\mathcal{E}}(h) = \hat{h}$, Lemma 4.19 guarantees that this twistor line is generic. In Section 5 we established invariance of the Hodge type of $c_2(\mathcal{A})$ under the twistor deformations defined by lines in \mathcal{M}_ϕ (Lemma 5.5), and we proved H -slope-stability of \mathcal{E} for $H = \pi_S^*h + \pi_M^*\hat{h}$ in Lemma 5.7. So, by Remark 5.4 and Theorem 5.2 we can extend \mathcal{A} over the first twistor family over the first line of the path. Further extensions over families $\mathcal{X}_2, \dots, \mathcal{X}_k$ exist automatically as the sheaf \mathcal{A}_t over $S_t \times M_t$ with trivial Picard group, $Pic(S_t) = Pic(M_t) = \langle 0 \rangle$, being slope-polystable with respect to some Kähler class (by Theorem 5.2), is automatically slope-polystable for any choice of Kähler class on $S_t \times M_t$ (see [19, Proposition 7.8]). The proof of Proposition 6.1 is complete. \square

6.2. Reduction to the case of a cyclic isometry. Here we prove Theorems 1.1 and 1.2 formulated in the introduction. Let us introduce a definition that is needed for formulating the proposition below.

Definition 6.2. Let S be a complex manifold. We say that a cohomology class $\alpha \in H^*(S, \mathbb{F})$, $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or \mathbb{C} , is *of analytic type* if it can be expressed as a polynomial of Chern classes of coherent sheaves on S .

Remark 6.3. Note that for a class $\alpha \in H^*(S, \mathbb{F})$, where S is a smooth projective variety, to be of analytic type is the same as to be algebraic.

Proposition 6.4. *Every rational Hodge isometry of Kähler $K3$ surfaces of cyclic type is of analytic type.*

Proof. Let X, Y be any $K3$ surfaces and $f : H^2(X, \mathbb{Q}) \rightarrow H^2(Y, \mathbb{Q})$ be a rational Hodge isometry. Up to replacing f with $-f$ we may regard f as a signed isometry. We want to prove that f is of analytic type by applying Proposition 6.1. In order to apply this proposition we need f to satisfy the Kähler cone condition from Definition 4.14, namely $f(K_X) \cap K_Y \neq \emptyset$. While this may not hold for an arbitrary f , in general there exists an element w in the Weyl group of Y , such that $w \circ f(K_X) \cap K_Y \neq \emptyset$, see Proposition 3.10 in [3, Ch. VIII]. This element w is a composition of isometries of the form

$$r_C : H^2(Y, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z}),$$

$$\alpha \mapsto \alpha - 2 \frac{(\alpha \cdot C)}{(C \cdot C)} C = \alpha + (\alpha \cdot C) C,$$

for some (-2) -curve C on Y . For each r_C the induced map of rational vector spaces $r_C : H^2(Y, \mathbb{Q}) \rightarrow H^2(Y, \mathbb{Q})$ maps the positive cone C_Y^+ to itself and so is a signed Hodge isometry. Thus the isometry w , as a composition of r_C 's, is a signed Hodge isometry as well. Denote by ψ the composition $w \circ f$. Choose any signed markings $\eta_X : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ and $\eta_Y : H^2(Y, \mathbb{Z}) \rightarrow \Lambda$ and set $\phi := \eta_Y \circ \psi \circ \eta_X^{-1}$. Then ψ is a signed Hodge isometry satisfying the Kähler cone condition. Thus the moduli space \mathcal{M}_ϕ is defined and we may now apply Proposition 6.1, according to which the quintuple $((X, \eta_X), (Y, \eta_Y), \psi)$ lies in the image of the forgetful map $\bar{p} : \overline{\mathcal{M}}_\phi \rightarrow \mathcal{M}_\phi$. This proves that ψ is of analytic type. Each isometry r_C is induced by the correspondence $\Delta + C \times C \in H^4(Y \times Y, \mathbb{Z})$, so each r_C is of analytic type. Thus w as a composition of r_C 's is of analytic type by Lemma 6.6 below, implying w^{-1} is of analytic type as well. Finally our original isometry $f = w^{-1} \circ \psi$ is of analytic type by Lemma 6.6. \square

Further by an isometry we mean an isometry of the rationalized $K3$ lattice $\Lambda_{\mathbb{Q}}$. The cyclic type isometries are not only simple looking ones, they are actually building blocks for all other rational isometries, that is, every rational isometry ϕ of $\Lambda_{\mathbb{Q}}$ is a composition of isometries of cyclic type. This follows from the following theorem.

Theorem 6.5. *Let F be a field of characteristic $\neq 2$ and V be a vector space over F with a non-degenerate symmetric bilinear form $q(\cdot, \cdot)$. Then the corresponding orthogonal group $O(V)$ is generated by reflections.*

For the proof of Theorem 6.5 see [8, Proposition 2.36].

For the proof of Theorem 1.2 we will need the following lemma.

Lemma 6.6. *Let $\alpha \in H^*(S_1 \times S_2, \mathbb{Q})$ and $\beta \in H^*(S_2 \times S_3, \mathbb{Q})$ be classes of analytic type for compact complex manifolds S_1, S_2, S_3 . Then the class $\pi_{13*}(\pi_{12}^*(\alpha) \cup \pi_{23}^*(\beta)) \in H^*(S_1 \times S_3, \mathbb{Q})$ is also of analytic type.*

Note that pull-back of a class of analytic type is a class of analytic type and product of two classes of analytic type is also a class of analytic type. So the only operation in the lemma for which preservation of analytic type is nontrivial is the pushforward. Preservation of analytic type under a pushforward (and, hence, Lemma 6.6) follows from a more general Lemma 6.7 formulated below. In order to formulate this lemma we need to introduce a new notation. Let X be a complex-analytic manifold. Denote by $Hdg(X)_{an}$ the subring of the cohomology ring $H^*(X, \mathbb{Q})$ generated by the classes of analytic type.

Lemma 6.7. *Let $f : X \rightarrow Y$ be a smooth morphism of compact complex-analytic manifolds X and Y . Then $f_*Hdg(X)_{an} \subset Hdg(Y)_{an}$.*

For the proof of Lemma 6.7 we refer to Section 7.

Proof of Theorem 1.2. Consider a general Hodge isometry $\varphi : H^2(S, \mathbb{Q}) \rightarrow H^2(S', \mathbb{Q})$ inducing via markings η, η' an isometry $\phi : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$. Up to change of sign \pm we may regard φ as a signed isometry. The isometry ϕ decomposes as a composition of cyclic type isometries, $\phi = \phi_k \circ \cdots \circ \phi_1$ by the Cartan-Dieudonne Theorem. Here we

may also assume that all ϕ_i are signed. Set $\eta_0 := \eta$ and $\eta_k := \eta'$. Starting with S and using the surjectivity of the period map for marked $K3$ surfaces we obtain a sequence of marked $K3$ surfaces

$$(S, \eta), (S_1, \eta_1) \dots, (S_{k-1}, \eta_{k-1}), (S', \eta'),$$

with periods $[\eta(\sigma_S)], l_i = \tilde{\phi}_i \circ \dots \circ \tilde{\phi}_1([\eta(\sigma_S)]), i = 1, \dots, k$. Note the equality $l_k = [\eta'(\sigma_{S'})]$. Set $\psi_i = \pm \eta_{i+1}^{-1} \phi_{i+1} \eta_i, i = 0, \dots, k-1$. Here we want the markings η_i to agree with the choice of the sign for ψ_i so that $((S_i, \eta_i), (S_{i+1}, \eta_{i+1}), \psi_i)$ belongs to \mathcal{M}_{ϕ_i} for all $0 \leq i \leq k-1$. Then we represent our φ as a composition $\varphi = \pm \psi_k \circ \dots \circ \psi_0$. Now, each ψ_i is of analytic type by Proposition 6.4 and is thus given by a class

$$\alpha_i \in H^{2,2}(S_i \times S_{i+1}, \mathbb{Q}),$$

of analytic type. Then taking the composition of the α_i 's as correspondences by Lemma 6.6 we obtain a class of analytic type

$$\alpha \in \oplus_p H^{p,p}(S_0 \times S_k, \mathbb{Q}),$$

which induces the isometry $\pm \varphi$. This completes the proof of Theorem 1.2. \square

Proof of Theorem 1.1. If S and S' were taken to be projective, then due to Remark 6.3, the isometry φ is algebraic. Thus Theorem 1.1 is proved. \square

7. APPENDIX

Proof of Lemma 6.7. We will prove the lemma by proving that the vector space $Hdg(X)_{an}$ is spanned over \mathbb{Q} by classes of the form $ch(\mathcal{F})td_f$, for \mathcal{F} a coherent analytic sheaf on X and td_f the Todd class of the relative tangent bundle of the morphism $f : X \rightarrow Y$. Then the statement will follow by Grothendieck-Riemann-Roch formula for complex-analytic manifolds, see [5],

$$f_*(ch(\mathcal{F}) \cdot td_f) = ch(\sum (-1)^i R^i f_*(\mathcal{F})),$$

and the Grauert theorem [9, Theorem 10.4.6], according to which the sheaves $R^i f_* \mathcal{F}$ occurring in the Grothendieck-Riemann-Roch formula are coherent analytic sheaves.

First of all note that the class td_f belongs to $Hdg(X)_{an}$ being an invertible element of this ring. Multiplication by td_f induces a linear automorphism of the vector space $Hdg(X)_{an}$. So it is sufficient to prove that $Hdg(X)_{an}$ is spanned over \mathbb{Q} by classes of the form $ch(\mathcal{F})$. In order to prove this we need two lemmas formulated below. Let V be the subring of $H^*(X, \mathbb{Q})$ generated by Chern characters $ch(\mathcal{F})$ of coherent analytic sheaves \mathcal{F} on X .

Lemma 7.1. *The subring V is spanned over \mathbb{Q} by classes of the form $ch(\mathcal{F})$.*

Lemma 7.2. *The subring V is equal to $Hdg(X)_{an}$.*

The idea of the proof of the first statement is based on multiplicativity of Chern characters of topological vector bundles with respect to tensor product,

$$ch(\mathcal{F}) \cdot ch(\mathcal{G}) = ch(\mathcal{F} \otimes \mathcal{G}),$$

for vector bundles \mathcal{F}, \mathcal{G} . Recall how the notion of Chern character extends from vector bundles to coherent analytic sheaves. Let $K_{an}(X)$ denote the K-ring generated by coherent analytic sheaves on X and $K_{top}(X)$ denote the K-ring generated by

topological vector bundles on X . It is clear how the Chern character is defined on the elements of $K_{top}(X)$. For a general complex manifold X a coherent analytic sheaf \mathcal{F} may not have a (global) resolution by locally free analytic sheaves, so we cannot define the Chern character on $K_{an}(X)$ in a straightforward way like we do it for $K_{top}(X)$. However, according to [2, Proposition 2.6] for a coherent analytic sheaf in $K_{an}(X)$ there exists a resolution of this sheaf by *real-analytic* vector bundles. Thus there is a map $h : K_{an}(X) \rightarrow K_{top}(X)$, sending the class of a coherent sheaf to the class of its real-analytic representative. So we define

$$ch([\mathcal{F}]) \stackrel{def}{=} ch(h([\mathcal{F}])),$$

where $[\mathcal{F}]$ means class of the sheaf \mathcal{F} in $K_{an}(X)$.

In general, when \mathcal{F} and \mathcal{G} are any coherent sheaves, though their tensor product as \mathcal{O}_X -modules is defined, we do not have the multiplicativity of the Chern character with respect to this tensor product. But we do have the equality

$$ch([\mathcal{F}]) \cdot ch([\mathcal{G}]) = ch([\mathcal{F}] \otimes [\mathcal{G}]),$$

which is a consequence of the analogous equality in $K_{top}(X)$ that we are going to explain below. The multiplicativity of the Chern character on $K_{top}(X)$ is a standard fact from topological K-theory, see, for example, [17, Chapter III, Corollary 11.17].

Let us first explain how tensor product in $K_{an}(X)$ is defined. Let a and b be classes in $K_{an}(X)$. If $\mathcal{U} = (\mathcal{U}_i, d_i)$ and $\mathcal{V} = (\mathcal{V}_i, d_i)$ are complexes of real analytic vector bundles representing respectively $h(a)$ and $h(b)$, then the complex of real analytic vector bundles $\mathcal{U} \otimes \mathcal{V}$ by definition of product in $K_{top}(X)$ represents $h(a) \otimes h(b) \in K_{top}(X)$. We recall that if

$$\mathcal{U} = \cdots \rightarrow U_{k+1} \xrightarrow{D_{k+1}^U} U_k \xrightarrow{D_k^U} \cdots,$$

and

$$\mathcal{V} = \cdots \rightarrow V_{k+1} \xrightarrow{D_{k+1}^V} V_k \xrightarrow{D_k^V} \cdots,$$

then the complex $\mathcal{U} \otimes \mathcal{V}$ has differentials D_k determined by their restrictions

$$D_{p+q}|_{U_p \otimes V_q} = D_p^U \otimes Id_{V_q} + (-1)^p Id_{U_p} \otimes D_q^V.$$

This is how tensor product $h(a) \otimes h(b) \in K_{top}(X)$ is defined. In order to define $a \otimes b$ for $K_{an}(X)$ we need to check that the product $h(a) \otimes h(b) \in K_{top}(X)$ belongs to the image of map h , $h(a) \otimes h(b) = h(c)$, $c \in K_{an}(X)$, so that

$$a \otimes b \stackrel{def}{=} c.$$

The fact that such c is correctly defined will be obtained in the course of proving the invariance of $Im h$ under tensor product.

Before starting the proof we need to make the following simplification. Every class in $K_{an}(X)$ or $K_{top}(X)$ is represented by a complex \mathcal{U} is equivalent to a class which is the alternating sum of the sheaf cohomology groups of the complex \mathcal{U} ,

$$[\mathcal{U}] \sim \oplus_i (-1)^i [\mathcal{H}^i(\mathcal{U})].$$

So, to define the product \otimes as a bilinear operation on the corresponding K-ring it suffices to define it for classes of sheaves. In our case, for a, b in $K_{an}(X)$ the classes $[\mathcal{H}^i(\mathcal{U})]$ correspond to coherent analytic sheaves $\mathcal{H}^i(\mathcal{U})$. So we may assume from the

beginning that the classes a, b in $K_{an}(X)$ are classes of complex-analytic sheaves A and B . Consider the product $[\mathcal{A}] \otimes [\mathcal{B}] \in K_{top}(X)$ defined by using real-analytic locally free resolutions \mathcal{A} and \mathcal{B} of sheaves A and B . Let us show that the resulting class belongs to the image of the map $h : K_{an}(X) \rightarrow K_{top}(X)$ by finding appropriate $c \in K_{an}(X)$ as above. Regarding the above simplification it is clear that for proving that $h(a) \otimes h(b)$ belongs to $Im\,h$ it is sufficient to show that cohomology sheaves $\mathcal{H}^k(\mathcal{A} \otimes \mathcal{B})$ are coherent analytic sheaves.

The notion of coherence and analyticity are local. But on a compact complex-analytic manifold for any coherent sheaf there exists, locally, a resolution by locally free analytic sheaves (holomorphic vector bundles). Take such local resolutions \mathcal{A}_{an} and \mathcal{B}_{an} for the sheaves A and B on a sufficiently small neighbourhood $W \subset X$. We then see that the sheaf $\mathcal{H}^k(\mathcal{A} \otimes \mathcal{B})|_W$ is isomorphic to sheaf $\mathcal{H}^k(\mathcal{A}_{an} \otimes \mathcal{B}_{an})$ (considered as a module over the sheaf of real-analytic complex valued functions) which is clearly a coherent analytic sheaf. The isomorphism follows from the fact that we have quasi-isomorphisms of complexes

$$(\mathcal{A} \otimes \mathcal{B})|_W \cong (A \otimes B)|_W \cong (\mathcal{A}_{an}(W) \otimes \mathcal{B}|_W) \cong \mathcal{A}_{an}(W) \otimes B|_W \cong \mathcal{A}_{an}(W) \otimes \mathcal{B}_{an}(W),$$

here A and B are regarded as complexes concentrated in degree zero. In this chain of isomorphisms we replace one of the factors at a time with a quasi-isomorphic complex, provided that the other factor is a free (and hence projective) resolution. Why such isomorphisms hold explained, for example, in [32, Chapter 2, Section 2.7]. So the product $h(a) \otimes h(b)$ indeed belongs to the image of the map h .

Note, that in the definitions of $Hdg(X)_{an}$ and V the Chern characters of coherent sheaves may be replaced by Chern characters of classes of $K_{an}(X)$, which will not change our V and $Hdg(X)_{an}$ as subsets of $H^*(X)$. So this proves that the ring V is spanned over \mathbb{Q} by Chern characters of coherent sheaves. This completes the proof of Lemma 7.1.

Now let us prove Lemma 7.2. For this it is sufficient to show that any Chern class $c_i(\mathcal{F})$ for \mathcal{F} a coherent analytic sheaf on X belongs to V . Actually, from the formula expressing the Chern character of \mathcal{F} as a polynomial of Chern classes of \mathcal{F} it follows that it is sufficient to prove that V contains together with every class $ch(\mathcal{F})$ all its homogeneous components. Let us prove this statement first for \mathcal{F} being a vector bundle and then extend the idea of the proof to the general case.

Let \mathcal{F} be a holomorphic vector bundle on X . By definition V contains Chern characters $ch(\mathcal{F}_k)$ of holomorphic vector bundles \mathcal{F}_k defined by induction,

$$\mathcal{F}_0 = \mathcal{F},$$

$$\mathcal{F}_k = \mathcal{F}_{k-1} \wedge \mathcal{F}_{k-1}, k \geq 1.$$

The Chern characters of \mathcal{F}_k and \mathcal{F}_{k-1} are related in the following way (the proof is given below):

$$(9) \quad ch(\mathcal{F}_k) = \frac{(ch(\mathcal{F}_{k-1}))^2}{2} - r_2 ch(\mathcal{F}_{k-1}),$$

where r_2 is the automorphism of the cohomology ring $H^*(X, \mathbb{Q})$ acting by multiplication by 2^i on $H^{2i}(X, \mathbb{Q})$.

Proof of Formula (9) for vector bundles. For Chern character $ch(\mathcal{F})$ we have the equality

$$ch(\mathcal{F}) = \sum_{i=1}^r e^{x_i}, r = rk(\mathcal{F}),$$

where the x_i 's are the Chern roots of \mathcal{F} . For further cohomological calculations we may assume, via the splitting principle, that \mathcal{F} is a direct sum of line bundles L_i with Chern characters e^{x_i} . Then for $ch(\mathcal{F} \wedge \mathcal{F})$ we have

$$ch(\mathcal{F} \wedge \mathcal{F}) = \sum_{i < j, i=1}^r e^{x_i + x_j} = \frac{ch(\mathcal{F})^2}{2} - \frac{r_2 ch(\mathcal{F})}{2},$$

where r_2 is the automorphism of the ring $H^*(X, \mathbb{Q})$ acting as multiplication by 2^j on $H^j(X, \mathbb{Q})$. \square

Formula (9) tells us, in particular, that the subring of the ring V , generated by the Chern characters of holomorphic vector bundles, is invariant under the automorphism r_2 . So V contains all the elements of the form

$$r_2^k ch(\mathcal{F}), 0 \leq k \leq n = \dim_{\mathbb{C}} X.$$

Now a simple argument involving nonvanishing of the corresponding Vandermonde determinant tells us that V contains all the homogeneous components of $ch(\mathcal{F})$.

Let us get to the general case when \mathcal{F} is a coherent analytic sheaf on X . Like previously, we may hope that Formula 9 extends from vector bundles to coherent analytic sheaves \mathcal{F} regarded as classes of $K_{an}(X)$. Then we would be able to proceed as in the case of vector bundles. So we need to define the wedge (self-) product for classes of $K_{an}(X)$. Again, like above, it is sufficient to define such wedge product for classes represented by coherent analytic sheaves. The definition of the wedge product involves the tensor product introduced above, so again we will be dealing with resolutions of coherent analytic sheaves by real-analytic vector bundles. The correctness is proved in a very similar way to how we proved the correctness of the tensor product for $K_{an}(X)$, that is, via existence of local resolutions by holomorphic vector bundles.

Again, every class in $K_{an}(X)$ is a class of a coherent analytic sheaf \mathcal{F} , for which, as earlier, there exists a resolution by real-analytic vector bundles V_k ,

$$h([\mathcal{F}]) = [\cdots \rightarrow V_{k+1} \rightarrow V_k \rightarrow \cdots] = [\bigoplus_{j \text{ even}} V_j] - [\bigoplus_{k \text{ odd}} V_k].$$

For classes in $K_{top}(X)$ of the form $[A] - [B]$ for vector bundles A, B we have the wedge self-product defined as in [1, Chapter III, §1], that is

$$([A] - [B]) \wedge ([A] - [B]) = [A \wedge A] - [A \otimes B] + [Sym^2 B].$$

Taking Chern character of both sides of this equality we get

$$\begin{aligned} ch(([A] - [B]) \wedge ([A] - [B])) &= ch(A \wedge A) - ch(A \otimes B) + ch(Sym^2 B) = \\ &= \frac{(ch(A))^2}{2} - \frac{r_2 ch(A)}{2} - ch(A)ch(B) + ch(Sym^2 B). \end{aligned}$$

Now the proof of Formula 9 in the general case reduces to the following lemma.

Lemma 7.3. *For a vector bundle B on a manifold X one has*

$$ch(Sym^2 B) = \frac{(ch(B))^2}{2} + r_2 \frac{ch(B)}{2}.$$

Indeed, then

$$ch([A] - [B]) \wedge ([A] - [B]) = \frac{(ch([A] - [B]))^2}{2} - \frac{r_2 ch([A] - [B])}{2},$$

which would prove Formula 9 in general case.

Proof of Lemma 7.3. We have

$$B \otimes B \cong B \wedge B \oplus Sym^2 B,$$

which together with above established Formula 9 for vector bundles proves the equality for $ch(Sym^2 B)$. \square

Finally, applying the Vandermonde determinant argument we obtain the proof of Lemma 7.2 and, hence, Lemma 6.7 in the general case. \square

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REFERENCES

- [1] Atiyah, M.F., *K-theory*, W.A. Benjamin, Inc, 1964.
- [2] Atiyah, M.F., Hirzebruch, F., *Analytic cycles on complex manifolds*, Topology Volume 1, Issue 1, January-March 1962, pp. 25-45.
- [3] Barth, W., Hulek, K., Peters, C., Ven, A. van de, *Compact complex surfaces*, 2nd ed. 1995, XII, 436 p.
- [4] Beauville, A., *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom., 18(4):pp. 755-782 (1984), 1983.
- [5] Brian, O', N.R., Toledo, D., Tong Y. L. L., *Grothendieck-Riemann-Roch for complex manifolds*, Bull. Amer. Math. Soc. (N.S.) Volume 5, Number 2 (1981), pp. 182-184.
- [6] Burns, D., Rapoport, M., *On the Torelli problem for kählerian K3 surfaces*, Ann. scient. Ec. Norm. Sup., 4^e serie, t. 8, 1975, p. 235-274.
- [7] Caldararu, A., *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Ph.D. Thesis, Cornell University, May 2000.
- [8] Gerstein, L. J., *Basic quadratic forms*, Graduate Studies in Math. Vol. 90, American Math. Soc. 2008.
- [9] Grauert, H., Remmert, R., *Coherent analytic sheaves*, Springer, 1984.
- [10] Hitchin, N. J., Karlhede, A., Lindström, U., Rocek, M., *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. Volume 108, Number 4 (1987), pp. 535-589.
- [11] Huybrechts, D., *A global Torelli theorem for hyper-Kähler manifolds [after M. Verbitsky]*, Astérisque No. 348 (2012), pp. 375-403.
- [12] Huybrechts, D., *Compact hyperkähler manifolds: basic results*, Invent. Math. 135(1999), no. 1, pp. 63-113.
- [13] Huybrechts, D., Lehn, M., *The geometry of moduli spaces of sheaves*, second edition, Cambridge University Press, 2010.
- [14] Huybrechts, D., *Erratum to the paper: Compact hyper-Kähler manifolds: basic results*, Invent. Math., 152 (2003), no. 1, pp. 209-212.

- [15] Huybrechts, D., Nigel R. O'Brian, Domingo Toledo, and Yue Lin L. Tong, *Compact hyperkähler manifolds* in M.Gross, D. Joyce and D. Huybrechts, *Calabi-Yau Manifolds and Related Geometries* Universitext 2003, pp. 161-225.
- [16] Lam, T. Y., *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, Volume 67, American Mathematical Society, 2004.
- [17] Lawson, H. Blaine, Michelson, Marie-Louise, *Spin geometry*, Princeton mathematical series 38, Princeton University Press, 1989.
- [18] Markman, E., *On the monodromy of moduli spaces of sheaves on K3 surfaces*, J. Algebraic Geom. 17 (2008), pp. 29-99.
- [19] Markman, E., *The Beauville-Bogomolov class as a characteristic class*, arXiv:1105.3223 [math.AG].
- [20] Markman, E., Mehrotra, S., *A global Torelli theorem for rigid hyperholomorphic sheaves*, arXiv:1310.5782v1 [math.AG].
- [21] Markman, E., *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Proceedings of the conference "Complex and Differential Geometry", Springer Proceedings in Mathematics, 2011, Volume 8, pp. 257-322.
- [22] Mukai, S., *On the moduli space of bundles on K3 surfaces, I*, Tata Institute of Fundamental Research, Bombay, January, 1984.
- [23] Mukai, S., *Duality of polarized K3 surfaces*, In: K. Hulek (ed.) et al., New trends in algebraic geometry, pp. 311-326, London Math.Soc. Lecture Note Ser. 264, Cambridge Univ. Press, Cambridge, 1999.
- [24] Mukai, S., *Vector bundles on a K3 surface*, Proceedings of the ICM, Beijing 2002, vol. 2, 495–502, available at arXiv:math/0304303v1 [math.AG].
- [25] Nikulin, V. V., *Integral symmetric bilinear forms and some of their applications*, 1980 Math. USSR Izv. **14** 103.
- [26] Nikulin, V. V., *On correspondences between surfaces of K3 type*, (in Russian), translation in: Math. USSR-Izv. **30** (1988), pp. 375-383.
- [27] Piatetski-Shapiro, I.I., Shafarevich, I.R., *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR: Izvestija 5 (1971), pp. 547-587.
- [28] Ramòn-Mari, J., *On the Hodge conjecture for products of certain surfaces*, Collect. Math. 59, 1 (2008), pp. 1-26.
- [29] Shafarevitch I.R., *Le theoreme de Torelli pour les surfaces algebriques de type K3*, Actes du Congres International des Mathematiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 413-417.
- [30] Verbitsky, M., *Hyperholomorphic sheaves and new examples of hyperkähler manifolds*, arXiv:alg-geom/9712012v2.
- [31] Verbitsky, M., *Mapping class group and a global Torelli theorem for hyperkähler manifolds*, Duke Math. J. Volume 162, Number 15 (2013), pp. 2929-2986.
- [32] Weibel, C., *Introduction to homological algebra*, Cambridge studies in advanced mathematics 38, Cambridge University Press, 2003.

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